DERIVATIVES OF THE GRAVITY FIELD OF A HOMOGENEOUS POLYHEDRAL BODY

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I shall refer here to the derivation of the formula for the gravity field of homogeneous polyhedral body presented in Pohánka (1988), hereafter referenced to as HPB. If D denotes the interior of the polyhedral body and ρ is the (difference) density within the body, then the gravity potential of the body at the point $\boldsymbol{r}$ is (formula 1 of HPB)

$$V(\boldsymbol{r}) = -\,\kappa\rho \int_D \mathrm{d}\tau' \,\frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|}, \tag{1}$$

where κ is the gravitational constant and $\mathrm{d}\tau'$ is the volume element at the point $\boldsymbol{r}'$. The intensity of the gravity field is (formula 2 of HPB)

$$\boldsymbol{E}(\boldsymbol{r}) = -\,\nabla_{\boldsymbol{r}} V(\boldsymbol{r}) \tag{2}$$

and the gradient of the intensity is

$$\boldsymbol{D}(\boldsymbol{r}) = \nabla_{\boldsymbol{r}} \boldsymbol{E}(\boldsymbol{r}) = -\,\nabla_{\boldsymbol{r}}\nabla_{\boldsymbol{r}} V(\boldsymbol{r}), \tag{3}$$

thus $\boldsymbol{D}(\boldsymbol{r})$ is a symmetric tensor.

Resulting formula for the intensity of the gravity field reads (formula 24 of HPB)

$$\boldsymbol{E}(\boldsymbol{r}) = -\,\kappa\rho \sum_{k=1}^{K} \boldsymbol{n}_k \sum_{l=1}^{L(k)} \phi\Big(u_{k,l}(\boldsymbol{r}), v_{k,l}(\boldsymbol{r}), w_{k,l}(\boldsymbol{r}), z_k(\boldsymbol{r})\Big), \tag{4}$$

where (formula 25 of HPB)

$$\phi(u,v,w,z) = w\,L(u,v,w,z) + 2\,z\,A(u,v,w,z), \tag{5}$$

(formula 38 of HPB)

$$L(u,v,w,z) = -\,\ln \frac{\sqrt{v^2+w^2+z^2} - v}{\sqrt{u^2+w^2+z^2} - u} \tag{6}$$

and (formula 28 of HPB)

$$A(u,v,w,z) = \arctan \frac{\sqrt{v^2+w^2+z^2} - v + z}{w} - \arctan \frac{\sqrt{u^2+w^2+z^2} - u + z}{w}. \tag{7}$$

Taking the gradient of the intensity of the gravity field we obtain after a lengthy calculation the formula

$$\boldsymbol{D}(\boldsymbol{r}) = \kappa\rho \sum_{k=1}^{K} \boldsymbol{n}_k \sum_{l=1}^{L(k)} \bigg(\boldsymbol{\nu}_{k,l} L\Big(u_{k,l}(\boldsymbol{r}), v_{k,l}(\boldsymbol{r}), w_{k,l}(\boldsymbol{r}), z_k(\boldsymbol{r})\Big) + 2\,\eta_k(\boldsymbol{r})\,\boldsymbol{n}_k A\Big(u_{k,l}(\boldsymbol{r}), v_{k,l}(\boldsymbol{r}), w_{k,l}(\boldsymbol{r}), z_k(\boldsymbol{r})\Big)\bigg), \tag{8}$$

where

$$\eta_k(\boldsymbol{r}) = \mathrm{sign}(\boldsymbol{n}_k \cdot (\boldsymbol{a}_{k,1} - \boldsymbol{r})) \tag{9}$$

and function sign(x) is defined as follows

$$\begin{aligned} x<0: &\quad \operatorname{sign}(x) = -1, \\ x=0: &\quad \operatorname{sign}(x) = 0, \\ x>0: &\quad \operatorname{sign}(x) = 1. \end{aligned} \tag{10}$$

Note that the expression (8) of the tensor $\boldsymbol{D}(\boldsymbol{r})$ is not explicitly symmetric, although this tensor is symmetric.

For further discussion we write formula (8) in the form

$$\boldsymbol{D}(\boldsymbol{r}) = \kappa\rho \sum_{k=1}^{K} \Bigg(\sum_{l=1}^{L(k)} \boldsymbol{n}_k \boldsymbol{\nu}_{k,l} L\Big(u_{k,l}(\boldsymbol{r}), v_{k,l}(\boldsymbol{r}), w_{k,l}(\boldsymbol{r}), z_k(\boldsymbol{r})\Big) + 2\,\eta_k(\boldsymbol{r})\, \boldsymbol{n}_k \boldsymbol{n}_k \sum_{l=1}^{L(k)} A\Big(u_{k,l}(\boldsymbol{r}), v_{k,l}(\boldsymbol{r}), w_{k,l}(\boldsymbol{r}), z_k(\boldsymbol{r})\Big)\Bigg). \tag{11}$$

Asymmetry of the expression for the tensor $\boldsymbol{D}(\boldsymbol{r})$ comes from the first term in big brackets. As any edge of a polyhedral body belongs to two sides of this body, any edge contributes to the total value of the tensor $\boldsymbol{D}(\boldsymbol{r})$ by two tensor terms and it can be shown that the sum of these two terms is a symmetric tensor. However, it is not necessary to write the formula (8) in an explicitly symmetric form; it is simpler to calculate by the numerical evaluation all 9 components of the tensor $\boldsymbol{D}(\boldsymbol{r})$ and then to symmetrize it.

For the numerical calculation we express function $L(u,v,w,z)$ according to Pohánka (1990) and Pohánka (1998), formula (53), in the form

$$\begin{aligned} \operatorname{sign}(u) = \operatorname{sign}(v): &\quad L(u,v,w,z) = \operatorname{sign}(v)\ \ln \frac{\sqrt{v^2+w^2+z^2}+|v|}{\sqrt{u^2+w^2+z^2}+|u|}, \\ \operatorname{sign}(u) \neq \operatorname{sign}(v): &\quad L(u,v,w,z) = \ln \frac{(\sqrt{v^2+w^2+z^2}+|v|)(\sqrt{u^2+w^2+z^2}+|u|)}{w^2+z^2}, \end{aligned} \tag{12}$$

and function $A(u,v,w,z)$ according to HPB, formula (33), and Pohánka (1998), formulae (39), (40), in the form

$$A(u,v,w,z) = -\ \arctan \frac{2\,w\,(v-u)}{T(u,v,w,z)^2 - (v-u)^2 + 2\,T(u,v,w,z)\,z}, \tag{13}$$

$$T(u,v,w,z) = \sqrt{u^2+w^2+z^2} + \sqrt{v^2+w^2+z^2}. \tag{14}$$

As it was shown in HPB (formula 41), function $L(u,v,w,z)$ is well defined for $w^2+z^2>0$. If $w^2+z^2 \to +0$, we distinguish two cases: if u and v are either both positive or both negative, function $L(u,v,w,z)$ is according to (12) well defined; if $u \leq 0$ and $v \geq 0$ (note that $u \leq v$ always), function $L(u,v,w,z)$ diverges logarithmically. Thus the first term in big brackets in (11) is singular when the point of calculation lies on some edge of the body.

Further, it was shown in HPB (formula 37) that the function $A(u,v,w,z)$ is well defined for $z>0$. If $z \to +0$, we see from (13) and (14) that the denominator in the arctangent funcion can be zero only if $T(u,v,w,z) = v-u$, and according to the triangle inequality this can happen only if $w=0$, $u \leq 0$ and $v \geq 0$. Therefore also the second term in big brackets in (11) is singular when the point of calculation lies on some edge of the body.

Another source of singularity is the function $\eta_k(\boldsymbol{r})$ in the second term in big brackets in (11); this function is discontinuous for $z_k(\boldsymbol{r})=0$, thus if the point of calculation lies in the plane of the

k-th side. It can be shown that for $z_k(\boldsymbol{r}) \to +0$ the sum in the second term in big brackets in (11) has the limit 0 if the point of calculation lies (in the plane of the k-th side) outside the side, and $-\pi$ if the point of calculation lies inside the side. In conclusion, the whole second term in big brackets in (11) is well defined for any point of calculation which does not lie in the k-th side of the body; for the point of calculation lying in this side, this term has the exterior (with respect to the body) limiting value $2\pi\,\boldsymbol{n}_k\boldsymbol{n}_k$ and interior limiting value $-2\pi\,\boldsymbol{n}_k\boldsymbol{n}_k$. This means that the tensor $\boldsymbol{D}(\boldsymbol{r})$ changes its value by $-4\pi\kappa\rho\,\boldsymbol{n}_k\boldsymbol{n}_k$ if the point of calculation crosses the surface of the body inwards (within some side). As the trace of the tensor $\boldsymbol{D}(\boldsymbol{r})$ is according to (3) equal to $-\Delta_{\boldsymbol{r}}V(\boldsymbol{r})$, we have the agreement with the Poisson equation

$$\Delta_{\boldsymbol{r}}V(\boldsymbol{r}) = 4\pi\kappa\rho(\boldsymbol{r}), \qquad (15)$$

because $\rho(\boldsymbol{r}) = \rho$ within the body and $\rho(\boldsymbol{r}) = 0$ outside the body.

For the numerical calculation we use the formula (8), but (similarly as in HPB) we replace the quantity z in the functions $L(u,v,w,z)$ and $A(u,v,w,z)$ by $z+\varepsilon$, where ε is a small positive number. Then the resulting formula is numerically safe for any position of the calculation point. However, this does not mean that we should not be cautious for the points in the vicinity of the surface of the body: we have to be sure whether we aim to calculate the exterior or interior value of the gradient of intensity, thus whether the point of calculation lies in the exterior or interior of the body. Furthermore, on the contrary to the formula for intensity considered in HPB, here the numerical error caused by introducing the quantity ε can be significant for the points in the vicinity of the surface of the body: whereas the correct value may be infinite (on some edge of the body), the calculated value will be always finite (depending on the value of parameter ε).

References

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