Solution of the inverse problem of gravimetry for a spherical planetary body using the decomposition of the interior potential into a series of polyharmonic functions

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Abstract: The interior potential of a spherical planetary body (and the corresponding density distribution) is expressed as the sum of two parts, the first given uniquely by the external gravity field and the surface density and the second depending on an arbitrary function. The first part of the potential (density distribution) is shown to be a 3-harmonic (biharmonic) function, while the second part can be expressed as a series whose i-th term (for $i \geq 0$) is an $(i+4)$-harmonic ($(i+3)$-harmonic) function. From this general solution a single solution is then chosen: this is done by imposing certain natural conditions on it, among others that this particular solution is an n-harmonic function for n as small as possible.

The paper explains shortly this method of solving the inverse problem of gravimetry; details are presented in *(Pohánka 1993)*.

1. Introduction

The inverse problem of gravimetry is mostly solved by introducing some model of the density function whose parameters are determined by the comparison of the measured and calculated gravity field using the method of the least squares. This approach is suitable if the model is relatively simple, otherwise the calculation of the parameters with sufficient accuracy can become extremely time-consuming. Moreover, the situation is complicated by the fact that the inverse problem of gravimetry has infinitely many solutions, and the calculation need not then lead to a satisfactory result. Therefore it would be advantageous to have some method allowing the density function to be calculated directly, without any prior knowledge of its model. Of course, any such method should take into account the existence of infinitely many density functions generating the given external gravity field. Apparently these two conditions can hardly be satisfied simultaneously, but this is not the case: such a method is presented in *(Pohánka 1993)* (hereinafter referred to as IPG). This work is rather mathematical (although not written strictly in mathematical style) and it is selfconsistent. As it has more than 200 pages, it could not be published in a journal and thus made known to the general public. The purpose of this paper is to explain shortly the new method, as it may be interesting for research workers in gravimetry and applied mathematics, and, perhaps even more, for people working in exploratory geophysics: the resulting formulae are directly applicable for calculation of the density from the gravity data.

The main idea of the new method is very simple: if the inverse problem of gravimetry has infinitely many solutions, is it possible to find among these solutions the simplest one? The answer can be given if we define what does it mean that the particular solution is the simplest: it is almost evident that it is the solution that requires the least information for its description. Of course, we are looking for an exact solution and thus it can hardly be represented by some standard model consisting of a number of simple bodies with a density constant inside each particular body and also in their surroundings. The simplicity can be expressed better as the smoothness of the density function in the domain where the density is unknown (the interior domain). It remains to define some exact

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measure of the smoothness: this can be done by choosing a functional of density whose value will represent this measure. The simplest density function can then be found by performing the variation of this functional (with respect to the density) under the condition that the gravity potential outside the interior domain is given (of course, it is sufficient to use the value of the potential at the boundary of the interior domain).

It may seem that the most natural form of this functional of density (expressing the smoothness of density) is the integral of the square of its gradient over the whole interior domain; in this case it can be easily shown that the density function determined by the minimum of this functional is biharmonic. Nevertheless, if this functional is defined as the integral of the square of the density, the resulting density is a harmonic function, and thus it is even simpler than in the previous case. It is well known that a harmonic function (in a compact domain with a sufficiently smooth boundary) is given uniquely by its boundary value; thus the amount of information needed for the description of a harmonic density function is the same as the amount of information of the input (the value of the potential at the boundary). As any biharmonic function can be constructed from two harmonic functions, in the case of the biharmonic density function we need some additional information for the description of some unique solution. This information can be (for example) the value of the density at the boundary of the interior domain.

It can be objected that such smooth density functions do not correspond in most cases to the real situation, as there are usually many disturbing bodies with more or less sharp boundaries (thus it would be better to look for the density in the form of a piecewise smooth function). Nevertheless, if there are some local anomalies of the gravity potential, no smooth solution of the inverse problem can be as smooth as to be constant in the whole interior domain (for the moment we neglect the effects caused by the form of the boundary of this domain), because a constant density cannot generate such a potential at the boundary. Thus there have to be some density inhomogeneities anyway; the question is whether they correspond to reality. Such a smooth solution can at least represent a zeroth approximation of the real density and thus can be used as the basis for the construction of more realistic models. If it were possible to describe any solution of the inverse problem (or at least any solution continuous in the interior domain) in some form, the construction of such models would be much easier; the restriction to the continuous functions is not too severe, as piecewise continuous functions can be approximated sufficiently well by the continuous functions.

The possibility of finding a closed formula expressing a harmonic function in some domain as a function of its boundary value (the Dirichlet problem) depends on the complexity of the boundary of this domain. Thus it is reasonable to treat the cases of the simple boundaries first. From the practical point of view it would be the best to begin with the simplest possible boundary: the plane dividing the lower halfspace (this is the interior domain) from the upper (where the density is assumed to be zero); the input is then the gravity potential (or its derivative in the direction normal to the plane) given for the whole plane. Although such a situation is almost realistic, it is disadvantageous from the mathematical point of view, as the domain where the density can be nonzero is infinite and there arise problems with asymptotic values of density at infinity. Therefore, it is much better to define the interior domain as the interior of a sphere; in this case this domain represents the interior of a planetary body (in the approximation of the absolutely smooth spherical surface). The inverse problem is treated in IPG for such a domain; it can be solved exactly and the method of its solution can serve as the starting point for solving the more general case of an arbitrary (smooth) surface of the planetary body.

It has to be emphasized that this choice of the interior domain does not imply that the derived formulae are suitable only for the global inverse problem: any local problem can be solved as well. Of course, if the input is given only on a (small) part of the surface, it has to be assumed that it is zero (or constant) elsewhere on the surface, and the density can then be calculated (with a certain reliability) only in some (small) domain lying immediately below the former part of the surface. If the dimensions of this part of the surface are much smaller than the radius of the body, it is possible to use the asymptotic version of the formulae that can be easily derived from the original one by letting the radius of the body tend to infinity (thus the planar version of the formulae) and to solve the inverse problem in this approximation (here again it is not necessary for the input to be given on the whole planar surface).

Now it is possible to present briefly the solution of the inverse problem of gravimetry derived in IPG. The exact formulation of the problem is as follows: the gravity field is assumed to be generated by the matter in the interior of a spherical body with radius R; the density of matter is assumed to be smooth (exact conditions are given in IPG, here it is sufficient to say that the density is a continuous function in the interior of the body and it has a continuous limit from the interior to the surface). At the surface of the body the value of the gravity potential or its normal derivative (with respect to the surface) is given (as the input), and the surface value of the density (equal to its limit from the interior) can also be given. The first problem is to find any density function (from the class defined by the conditions mentioned above) generating the given external gravity field and (if also the surface value of the density is given) having this surface value; these density functions represent the class described below as the general solution of the inverse problem. The second problem is to choose one particular density function (from this general solution) which can serve as a representative of the whole class of solutions; the exact conditions for this choice can be formulated only after the first problem is solved.

2. General solution of the inverse problem

We use the spherical coordinate system (r, ϑ, φ) with the origin in the centre of the body. The function of spherical coordinates can be expressed as $f(r, \vartheta, \varphi)$, or briefly as $f(r, \boldsymbol{n})$, where $\boldsymbol{n}$ is the unit vector

$$\boldsymbol{n} = \boldsymbol{i} \sin\vartheta \cos\varphi + \boldsymbol{j} \sin\vartheta \sin\varphi + \boldsymbol{k} \cos\vartheta \tag{2.1}$$

and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are mutually orthogonal constant unit vectors; further we denote for brevity

$$\int \mathrm{d}\Omega \; f(r, \boldsymbol{n}) = \int_0^\pi \mathrm{d}\vartheta \;\sin\vartheta \int_0^{2\pi} \mathrm{d}\varphi \; f(r, \vartheta, \varphi). \tag{2.2}$$

Spherical functions $\mathrm{Y}_{n,m}(\boldsymbol{n})$ (they are nonzero only for $n \geq |m|$) are defined as

$$\mathrm{Y}_{n,m}(\boldsymbol{n}) = \sqrt{(2n+1)\frac{(n-|m|)!}{(n+|m|)!}}\; \mathrm{P}_n^{|m|}(\cos\vartheta)\, \mathrm{e}^{im\varphi}, \tag{2.3}$$

where $\mathrm{P}_n^m(u)$ are the associated Legendre functions; hence,

$$\mathrm{Y}^*_{n,m}(\boldsymbol{n}) = \mathrm{Y}_{n,-m}(\boldsymbol{n}). \tag{2.4}$$

The spherical functions form a complete orthonormal system (on the surface of the unit sphere): for $n \geq |m|, n' \geq |m'|$

$$\int \mathrm{d}\Omega \; \mathrm{Y}^*_{n,m}(\boldsymbol{n})\; \mathrm{Y}_{n',m'}(\boldsymbol{n}) = 4\pi\, \delta_{n,n'}\, \delta_{m,m'}. \tag{2.5}$$

It is first shown that any function sufficiently smooth in the closed domain $r \leq R$ can be expressed in the form

$$f(r, \boldsymbol{n}) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} \sum\nolimits_{k\geq 0} f_{k,n,m}\, r^{n+2k}\, \mathrm{Y}_{n,m}(\boldsymbol{n}), \tag{2.6}$$

where the series converges uniformly and absolutely in this domain and (as this function is real) coefficients $f_{k,n,m}$ satisfy the condition

$$f^*_{k,n,m} = f_{k,n,-m}. \tag{2.7}$$

It is suitable to introduce the normalized radial coordinate s

$$s = r/R \tag{2.8}$$

(then in the interior domain $0 \leq s < 1$, and on the surface of the body $s = 1$); if we define

$$f_{n,m}(\xi) = R^n \sum\nolimits_{k\geq 0} f_{k,n,m}\, R^{2k}\, \xi^k, \tag{2.9}$$

formula (2.6) becomes

$$f(r, \boldsymbol{n}) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} f_{n,m}(s^2)\, s^n\, \mathrm{Y}_{n,m}(\boldsymbol{n}). \tag{2.10}$$

The summation signs are defined as follows: $\sum_{n\geq k}$ ($\sum_{n\leq k}$) is the summation over n (the first variable in the condition) from k to ∞ (from $-\infty$ to k), and $\sum_{k\leq n\leq l}$ is the summation over n (the middle variable in the condition) from k to l if $k \leq l$, and zero otherwise ($\sum_{|n|\leq k}$ is the abbreviation of $\sum_{-k\leq n\leq k}$).

The potential of the gravity field $V(r, \boldsymbol{n})$ and the density of the matter $\rho(r, \boldsymbol{n})$ outside the body (for $r > R$) satisfy the equations

$$\rho(r, \boldsymbol{n}) = 0 \tag{2.11}$$

$$\Delta V(r, \boldsymbol{n}) = 0, \tag{2.12}$$

and the potential tends to zero at infinity; within the body (for $0 \leq r < R$) these two functions satisfy the Poisson equation

$$\Delta V(r, \boldsymbol{n}) = 4\pi\, \kappa\, \rho(r, \boldsymbol{n}), \tag{2.13}$$

where κ is the gravitational constant. As the potential and its gradient are continuous in the whole space, their limits on the surface of the body from inside and outside have to be the same; in the case of density its limit from inside is the surface density:

$$\rho(R, \boldsymbol{n}) = \lim\nolimits_{r\to R-} \rho(r, \boldsymbol{n}). \tag{2.14}$$

If we write the potential and density as a series of the spherical functions

$$V(r, \boldsymbol{n}) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} V_{n,m}(r)\, \mathrm{Y}_{n,m}(\boldsymbol{n}) \tag{2.15}$$

$$\rho(r, \boldsymbol{n}) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} \rho_{n,m}(r)\, \mathrm{Y}_{n,m}(\boldsymbol{n}) \tag{2.16}$$

and use a suitable expression for the Laplace operator, we get the equations for the coefficients $V_{n,m}(r)$ and $\rho_{n,m}(r)$: for $r > R$

$$(\delta_r - n)(\delta_r + n + 1)\, V_{n,m}(r) = 0, \tag{2.17}$$

while for $0 \le r < R$

$$(\delta_r - n)(\delta_r + n + 1)\, V_{n,m}(r) = 4\pi\, \kappa\, r^2\, \rho_{n,m}(r), \tag{2.18}$$

where δ_x is the differential operator defined as

$$\delta_x = x\, \partial_x \tag{2.19}$$

(∂_x is the partial derivative with respect to variable x). The solution of equation (2.17) tending to zero at infinity can be expressed as

$$V_{n,m}(r) = -\,\frac{1}{R}\, v_{n,m}\, s^{-n-1}, \tag{2.20}$$

where $v_{n,m}$ are constant coefficients. Within the body the coefficients of potential and density can be expressed according to (2.10) as

$$V_{n,m}(r) = -\,\frac{1}{R}\, u_{n,m}(s^2)\, s^n \tag{2.21}$$

$$\rho_{n,m}(r) = \frac{1}{4\pi\, \kappa\, R^3}\, \sigma_{n,m}(s^2)\, s^n \tag{2.22}$$

and inserting them in (2.18) yields the equation (holding for $0 \le \xi < 1$)

$$-\,2\, \partial_\xi (2\delta_\xi + 2n + 1)\, u_{n,m}(\xi) = \sigma_{n,m}(\xi). \tag{2.23}$$

The continuity of the potential and its gradient at the surface of the body implies the conditions

$$\lim_{s\to 1-} u_{n,m}(s^2)\, s^n = \lim_{s\to 1+} v_{n,m}\, s^{-n-1} \tag{2.24}$$

$$\lim_{s\to 1-} \delta_s u_{n,m}(s^2)\, s^n = \lim_{s\to 1+} \delta_s v_{n,m}\, s^{-n-1}, \tag{2.25}$$

which after some algebra yield

$$u_{n,m}(1) = v_{n,m} \tag{2.26}$$

$$[(2\delta_\xi + 2n + 1)\, u_{n,m}(\xi)]_{\xi=1} = 0, \tag{2.27}$$

where $[f(\xi)]_{\xi=1}$ denotes the limit of $f(\xi)$ for $\xi \to 1-$. We see that the first formula expresses the connection between the potential in the interior and exterior of the body, whereas the second formula represents a condition imposed solely on the interior potential. If we denote

$$w_{n,m}(\xi) = \partial_\xi^2 u_{n,m}(\xi), \tag{2.28}$$

we can easily derive the formula

$$u_{n,m}(\xi) = v_{n,m} + \frac{1}{2}\, (2n+1)\, (1-\xi)\, v_{n,m} + \int_\xi^1 \mathrm{d}\xi'\, (\xi' - \xi)\, w_{n,m}(\xi') \tag{2.29}$$

expressing the coefficients of the interior potential in a form automatically satisfying conditions (2.26) and (2.27) for any choice of coefficients $w_{n,m}(\xi)$ (provided the latter are continuous functions of ξ for $0 \le \xi \le 1$). Formula (2.23) then yields

$$\sigma_{n,m}(\xi) = (2n+1)(2n+3)\, v_{n,m} - \,4\, \xi\, w_{n,m}(\xi) + 2(2n+3) \int_\xi^1 \mathrm{d}\xi'\, w_{n,m}(\xi') \tag{2.30}$$

and we see that it is possible to express coefficients $u_{n,m}(\xi)$ and $\sigma_{n,m}(\xi)$ as a sum of two parts, one dependent on and the other independent of input coefficients $v_{n,m}$. If also the surface density is known, i.e. the values of coefficients $\sigma_{n,m}(\xi)$ for $\xi \to 1-$, we denote

$$z_{n,m}(\xi) = \partial_\xi w_{n,m}(\xi), \tag{2.31}$$

so that

$$w_{n,m}(\xi) = w_{n,m}(1) - \int_\xi^1 \mathrm{d}\xi' \; z_{n,m}(\xi'). \tag{2.32}$$

The values of $w_{n,m}(1)$ can be determined from the values of $\sigma_{n,m}(1)$ using formula (2.30)

$$w_{n,m}(1) = \frac{1}{4} \Big((2n{+}1)(2n{+}3) \, v_{n,m} - \sigma_{n,m}(1) \Big) \tag{2.33}$$

and formulae (2.29), (2.30) become

$$\begin{aligned} u_{n,m}(\xi) = {} & v_{n,m} + \frac{1}{2} \, (2n{+}1) \, (1-\xi) \, v_{n,m} + \frac{1}{8} \, \Big((2n{+}1)(2n{+}3) \, v_{n,m} - \sigma_{n,m}(1) \Big) \, (1-\xi)^2 - \\ & - \frac{1}{2} \int_\xi^1 \mathrm{d}\xi' \; (\xi' - \xi)^2 \, z_{n,m}(\xi') \end{aligned} \tag{2.34}$$

$$\begin{aligned} \sigma_{n,m}(\xi) = {} & \sigma_{n,m}(1) + \frac{1}{2} \, (2n{+}5) \, \Big((2n{+}1)(2n{+}3) \, v_{n,m} - \sigma_{n,m}(1) \Big) \, (1-\xi) + \\ & + 4 \, \xi \int_\xi^1 \mathrm{d}\xi' \; z_{n,m}(\xi') - 2(2n{+}3) \int_\xi^1 \mathrm{d}\xi' \; (\xi' - \xi) \; z_{n,m}(\xi'). \end{aligned} \tag{2.35}$$

Coefficients $\sigma_{n,m}(\xi)$ have the correct limit for $\xi \to 1-$ independently of the form of coefficients $z_{n,m}(\xi)$ (again provided the latter are continuous functions of ξ for $0 \le \xi \le 1$). We see that also in this case coefficients $u_{n,m}(\xi)$ and $\sigma_{n,m}(\xi)$ are expressed as a sum of two parts, one dependent on and the other independent of input coefficients $v_{n,m}$, $\sigma_{n,m}(1)$.

The formulae expressing potential $V(r, \boldsymbol{n})$ and density $\rho(r, \boldsymbol{n})$ in the interior of the body can be obtained by inserting expressions (2.29), (2.30) or (2.34), (2.35) into formulae (2.21), (2.22) and these into (2.15), (2.16). Coefficients $v_{n,m}$ can be expressed in terms of the surface value of potential $V(R, \boldsymbol{n})$ or (what is more usual) the surface value of its vertical gradient $g(R, \boldsymbol{n})$, where

$$g(r, \boldsymbol{n}) = \partial_r V(r, \boldsymbol{n}); \tag{2.36}$$

coefficients $\sigma_{n,m}(1)$ can be expressed in terms of the surface value of density $\rho(R, \boldsymbol{n})$. In view of formula (2.10) coefficients $w_{n,m}(\xi)$ (or $z_{n,m}(\xi)$) can be considered as the coefficients of decomposition of a certain function defined in the interior of the body and on its surface, and they can be expressed in terms of this function. This function is arbitrary (apart from the condition that it has to be continuous) and it represents the source of nonuniqueness of the solution of the considered inverse problem. In this manner it is possible to obtain the formulae expressing the interior potential $V(r, \boldsymbol{n})$ and density $\rho(r, \boldsymbol{n})$ as an integral transformation of the input and of an arbitrary function; these formulae are presented in IPG for four possible cases of input: with and without the knowledge of the surface density $\rho(R, \boldsymbol{n})$ and the knowledge of either surface potential $V(R, \boldsymbol{n})$ or its vertical gradient $g(R, \boldsymbol{n})$.

For brevity only the formula expressing density $\rho(r, \boldsymbol{n})$ in the case that the input is represented by functions $g(R,\boldsymbol{n})$ and $\rho(R,\boldsymbol{n})$ will be presented here. The density in the interior of the body (and on its surface) can be expressed as the sum of two parts,

$$\rho(r,\boldsymbol{n}) = \rho_{\rm e}(r,\boldsymbol{n}) + \rho_{\rm i}(r,\boldsymbol{n}), \tag{2.37}$$

where the first part (index e stands for external)

$$\rho_{\rm e}(r,\boldsymbol{n}) = \frac{1}{4\pi}\int {\rm d}\Omega' \left(G_0^0(s,\boldsymbol{n}\cdot\boldsymbol{n}') - (1-s^2)\; G_1^2(s,\boldsymbol{n}\cdot\boldsymbol{n}')\right) \rho(R,\boldsymbol{n}') + \\ + \frac{6}{\pi\,\kappa\,R}\,(1-s^2)\,\frac{1}{4\pi}\int {\rm d}\Omega'\; D_3^0(s,\boldsymbol{n}\cdot\boldsymbol{n}')\; g(R,\boldsymbol{n}') \tag{2.38}$$

is determined (uniquely) by the external gravity field and the surface density, while the second part (index i stands for internal)

$$\rho_{\rm i}(r,\boldsymbol{n}) = -\frac{2}{\pi\,\kappa\,R^2}\,\lim\nolimits_{q\to 1-}\int_s^1 {\rm d}s'\; s'\;\frac{1}{4\pi}\int {\rm d}\Omega' \left(s^2\, G_0^0(qs/s',\boldsymbol{n}\cdot\boldsymbol{n}') - \right. \\ \left. -\,(s'^2 - s^2)\; G_1^1(qs/s',\boldsymbol{n}\cdot\boldsymbol{n}')\right)\; Z(Rs',\boldsymbol{n}') \tag{2.39}$$

is determined (uniquely) by an arbitrary (smooth) function $Z(r,\boldsymbol{n})$ defined in the interior of the body:

$$Z(r,\boldsymbol{n}) = -\frac{1}{R}\sum\nolimits_{n\ge 0}\sum\nolimits_{|m|\le n} z_{n,m}(s^2)\; s^n\; {\rm Y}_{n,m}(\boldsymbol{n}). \tag{2.40}$$

The integral kernels in formulae (2.38) and (2.39) are defined as

$$G_k^a(\lambda,\mu) = \sum\nolimits_{n\ge 0}(2n{+}1)\binom{n{+}a{+}k{-}1/2}{k}\;\lambda^n\;{\rm P}_n(\mu) \tag{2.41}$$

$$D_k^a(\lambda,\mu) = \sum\nolimits_{n\ge 0}\frac{2n{+}1}{n{+}1}\binom{n{+}a{+}k{-}1/2}{k}\;\lambda^n\;{\rm P}_n(\mu); \tag{2.42}$$

function $G_0^0(s,\boldsymbol{n}\cdot\boldsymbol{n}')$ is the Green function of the Dirichlet problem in the interior of a spherical domain. These integral kernels can be written in a closed form: particularly

$$G_1^a(\lambda,\mu) = 4\,A_2(\lambda,\mu) + 2(a-2)\,A_1(\lambda,\mu) - (a-1/2)\,A_0(\lambda,\mu) \tag{2.43}$$

$$G_k^0(\lambda,\mu) = \sum\nolimits_{0\le l\le k+1}\binom{l-3/2}{l}\,\frac{2l-3(k{+}1)}{l-3/2}\;A_{k+1-l}(\lambda,\mu) \tag{2.44}$$

$$D_k^0(\lambda,\mu) = \sum\nolimits_{0\le l\le k+1}\binom{l-3/2}{l}\,\frac{2l-3(k{+}1)}{l-3/2}\;B_{k+1-l}(\lambda,\mu), \tag{2.45}$$

where $(n\ge 0)$

$$A_n(\lambda,\mu) = \frac{{\rm P}_n(c(\lambda,\mu))}{d(\lambda,\mu)^{n+1}} \tag{2.46}$$

$$B_0(\lambda,\mu) = \frac{1}{\lambda}\,\ln\frac{d(\lambda,\mu)+1+\lambda}{d(\lambda,\mu)+1-\lambda} \tag{2.47}$$

$$B_{n+1}(\lambda,\mu) = \frac{1}{n+1} \sum\nolimits_{0\le l\le n} A_l(\lambda,\mu) \tag{2.48}$$

and

$$d(\lambda,\mu) = \sqrt{1-2\lambda\mu+\lambda^2} \tag{2.49}$$

$$c(\lambda,\mu) = \frac{1-\lambda\mu}{d(\lambda,\mu)}. \tag{2.50}$$

It has to be emphasized that density $\rho_{\rm e}(r,\boldsymbol{n})$ generates the given exterior gravity field with the surface value of vertical gradient $g(R,\boldsymbol{n})$ and has the given surface value $\rho(R,\boldsymbol{n})$, while density $\rho_{\rm i}(r,\boldsymbol{n})$ generates the zero exterior gravity field (with zero surface value of the vertical gradient) and has a zero surface value.

However, formulae (2.38), (2.39) are not quite convenient for the practical calculation of the interior density: the first holds only for $|\boldsymbol{r}| < R$ and in the region near the surface (which is the most important in practice) problems caused by the singularity of the integral kernels can arise; the second formula holds for $|\boldsymbol{r}| \le R$, but it contains a limit (that cannot be removed because of the same singularity) and thus it is almost unusable. Therefore, it was necessary to find a form of these integral formulae where the integral kernels do not contain unbounded terms. This was achieved by expressing the integral with kernel $K(s,\boldsymbol{n}\cdot\boldsymbol{n}')$ in the form

$$\frac{1}{4\pi}\int \mathrm{d}\Omega'\ K(s,\boldsymbol{n}\cdot\boldsymbol{n}')\ f(\boldsymbol{n}') = \frac{1}{2}\int_0^\pi \mathrm{d}\Theta\ \sin\Theta\ K(s,\cos\Theta)\ \Sigma(\boldsymbol{n},\cos\Theta)\ f(*), \tag{2.51}$$

where operator $\Sigma(\boldsymbol{n},\mu)$ is defined by the formula

$$\Sigma(\boldsymbol{n},\cos\Theta)\ f(*) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\Phi\ f((\boldsymbol{i_n}\cos\Phi + \boldsymbol{j_n}\sin\Phi)\sin\Theta + \boldsymbol{n}\cos\Theta) \tag{2.52}$$

and $\boldsymbol{i_n}$, $\boldsymbol{j_n}$ are two unit vectors that are orthogonal mutually and also to vector $\boldsymbol{n}$. This means that the integration over the circle on the unit sphere, whose centre is the point of the sphere given by vector $\boldsymbol{n}$ and whose angular radius viewed from the centre of the sphere is Θ, is performed first, followed by the integration with respect to angle Θ. The latter integration is performed using a suitable substitution for variable Θ to remove the unbounded terms in function $K(\lambda,\mu)$. This is done in IPG for all kernels appearing there (this is rather complicated); here we present the resulting formulae for the density. Introducing operator

$$\Sigma_1(\boldsymbol{n},\mu)\ f(*) = \frac{1}{\mu-1}\ (\Sigma(\boldsymbol{n},\mu)\ f(*) - f(\boldsymbol{n})), \tag{2.53}$$

defined for a smooth function $f(\boldsymbol{n})$ also for $\mu = 1$ (as the limit), we obtain

$$\begin{aligned}
\rho_{\rm e}(r,\boldsymbol{n}) = {} & \frac{1}{2}\int_{-1}^{1} \mathrm{d}\tau\ \left(3s^2 - \frac{3}{2}\ (1+s\tau)^2\right)\ \Sigma(\boldsymbol{n},a(s,\tau))\ \rho(R,*) + \\
& + \frac{3}{4\pi\,\kappa\,R}\ (1-s^2)\ \Bigg[\ \frac{5}{2}\ g(R,\boldsymbol{n}) + \\
& \qquad + \frac{1}{2}\int_{-1}^{1} \mathrm{d}\sigma\ \left(\frac{1}{2}\ \Delta(s,\sigma)\ (1-s\sigma)^2 + \frac{1}{3}\ (3+17s^2)\right) \cdot \\
& \qquad\qquad \cdot\ e(s,\sigma)\ \Sigma_1(\boldsymbol{n},b(s,\sigma))\ g(R,*) + \\
& \qquad + \frac{1}{2}\int_{-1}^{1} \mathrm{d}\tau\ (1+11s^2-5(1+s\tau)^2)\ \cdot \\
& \qquad\qquad \cdot\ h(s,\tau)\ \Sigma_1(\boldsymbol{n},a(s,\tau))\ g(R,*)\Bigg]
\end{aligned} \tag{2.54}$$

and

$$\rho_{\rm i}(r,\boldsymbol{n}) = -\frac{3}{\pi\,\kappa\,R^2}\int_s^1 {\rm d}s'\;s'\;\frac{1}{2}\int_{-1}^1 {\rm d}\tau\;(2s^2-(s'+s\tau)^2)\;\Sigma(\boldsymbol{n},a(s/s',\tau))\;Z(Rs',*), \tag{2.55}$$

where

$$\Delta(\lambda,\sigma) = 2-\frac{1-\lambda\sigma}{\lambda}\ln\frac{2+\lambda(1-\sigma)}{2-\lambda(1+\sigma)} \tag{2.56}$$

$$e(\lambda,\sigma) = \frac{1}{2}\,\frac{1-\sigma}{1-\lambda\sigma}\left(1+\frac{1-\lambda}{1-\lambda\sigma}\right) \tag{2.57}$$

$$h(\lambda,\tau) = \frac{1}{2}\,\frac{1-\tau}{1+\lambda}\left(1+\frac{1+\lambda\tau}{1+\lambda}\right) \tag{2.58}$$

$$a(\lambda,\tau) = \frac{\lambda+\tau}{1+\lambda\tau}+\frac{\lambda}{2}\left(1-\left(\frac{\lambda+\tau}{1+\lambda\tau}\right)^2\right) \tag{2.59}$$

$$b(\lambda,\sigma) = \sigma+\frac{\lambda}{2}\,(1-\sigma^2). \tag{2.60}$$

These formulae allow the density to be calculated at any point in the interior of the body, or on its surface. Therefore, it can be stated that the general solution of the inverse problem for a spherical body has been found.

3. Choice of the most suitable particular solution

Now we can treat this solution from the viewpoint of the degree of its harmonicity. A function $f(\boldsymbol{r})$ satisfying the condition

$$\Delta^n f(\boldsymbol{r}) = 0 \tag{3.1}$$

in some domain (here it is $|\boldsymbol{r}| < R$) is called n-harmonic (in this domain). The least natural number n such that the given function $f(\boldsymbol{r})$ satisfies this condition can be called the degree of harmonicity (of this function). It can be easily shown that if function $f(\boldsymbol{r})$ $(= f(r,\boldsymbol{n}))$ can be expressed in form (2.10), it is n-harmonic if, and only if, coefficients $f_{n,m}(\xi)$ are polynomials of ξ of degree $n-1$ at the most. Particularly, this function is harmonic (or 1-harmonic) if, and only if, coefficients $f_{n,m}(\xi)$ do not depend on ξ. As a consequence, any such n-harmonic function can be expressed (for $r < R$) as

$$f(r,\boldsymbol{n}) = \textstyle\sum_{1\le i\le n}(1-s^2)^{i-1}\,h_i(r,\boldsymbol{n}), \tag{3.2}$$

where $h_i(r,\boldsymbol{n})$ are functions harmonic in the same domain. As a harmonic function is uniquely determined by its boundary value, an n-harmonic function can be uniquely determined by n functions defined on this boundary (in our case on the surface of the body). Therefore, the degree of harmonicity of a function characterizes the amount of information required to describe this function.

In view of (2.15), (2.16), (2.21), (2.22), (2.34) and (2.35) it can be concluded that the part of the potential (density) dependent on the input is in general a 3-harmonic (biharmonic) function, although for a special value of the surface density it can even be biharmonic (harmonic). It can be easily shown that the part of the potential (density) independent of the input (if this part is nonzero) is not an n-harmonic function for any $n < 4$ $(n < 3)$. If coefficients $z_{n,m}(\xi)$ can be expressed as a power series of variable $1-\xi$, this part of the potential (density) becomes a series whose i-th term (for $i \geq 0$) is an $(i+4)$-harmonic ($(i+3)$-harmonic) function.

The degree of harmonicity of some part of the potential and density can be simply determined also if these are expressed in integral form: it can be easily shown that the function expressed by the integral on the l.h.s. of (2.51) is a harmonic function (for $r < R$) if its kernel $K(\lambda, \mu)$ is any one of the functions $G_k^a(\lambda, \mu)$, $D_k^a(\lambda, \mu)$. According to formulae (2.38) and (3.2) we can again conclude that function $\rho_\mathrm{e}(r, \boldsymbol{n})$ is biharmonic. Therefore, this function can be considered as the simplest (and smoothest) density generating the given external gravity field and having the given surface value. However, with respect to the nonuniqueness of the solution of the inverse problem there arises the question whether this density is able to yield some useful information about the real density distribution.

This question can be answered by considering a model situation where the only source of the gravity field is a single spherical inhomogenity (located under the surface of the body) with a constant (difference) density. The (external) gravity field generated by such a density distribution (it is a field of a point source located in the centre of the inhomogeneity) is the input for the solution of the inverse problem and the particular calculated density can be compared with the given one. These two densities can be very different, but it is natural to expect that the calculated density should have at least a local extremum in (or near to) the centre of the inhomogeneity.

Evidently, if the particular solution is a harmonic function, this requirement cannot be satisfied, as a harmonic function can have local extrema only at the boundary (i.e. on the surface of the body). It is shown that density $\rho_\mathrm{e}(r, \boldsymbol{n})$ does not satisfy this requirement either: for the near subsurface inhomogeneity (this case is the most interesting one) this density has a local extremum whose depth is approximately one third of the depth of the centre of this inhomogeneity. It is clear that such a solution of the inverse problem can hardly be useful in practice.

On the other hand, it is also hardly possible to consider the whole class of solutions; therefore, it would be advantageous to have some particular solution that could serve (in some sense) as the representative of all other solutions. This particular solution should be as close to reality as possible and also as simple as possible; in IPG such a particular solution is called the characteristic solution (of the inverse problem).

As density $\rho_\mathrm{e}(r, \boldsymbol{n})$ is, in view of (2.38), a linear integral transformation of the input, the same is required of the characteristic density $\rho_\mathrm{c}(r, \boldsymbol{n})$. This means that in formula (2.35) coefficients $z_{n,m}(\xi)$ are assumed to be linearly dependent on input coefficients $v_{n,m}$, $\sigma_{n,m}(1)$. Further, coefficients $z_{n,m}(\xi)$ are assumed to be polynomials of variable $1 - \xi$ such that the degree of harmonicity of function $Z(r, \boldsymbol{n})$ is N (the degree of harmonicity of the corresponding density is then $N + 2$). The model situation described above is then considered: it is required that the main extremum of the calculated density is identical with the centre of the given inhomogeneity and the minimal value of N satisfying this condition is sought. It is shown that this minimal value of N is 2, thus the characteristic density is a 4-harmonic function. The calculation of the characteristic solution is lengthy and, as the above requirements do not allow it to be determined uniquely, certain additional assumptions are made (conforming to the criterion of maximal simplicity).

We present here the formulae expressing density $\rho_\mathrm{c}(r, \boldsymbol{n})$ as the linear integral transformation of input functions $g(R, \boldsymbol{n})$ and $\rho(R, \boldsymbol{n})$. In analogy with (2.38) we have

$$\rho_\mathrm{c}(r, \boldsymbol{n}) = \frac{1}{4\pi} \int \mathrm{d}\Omega' \; Eb(s, \boldsymbol{n}\cdot\boldsymbol{n}') \; \rho(R, \boldsymbol{n}') + + \frac{1}{4\pi\,\kappa\,R} \, \frac{1}{4\pi} \int \mathrm{d}\Omega' \; Db(s, \boldsymbol{n}\cdot\boldsymbol{n}') \; g(R, \boldsymbol{n}'), \qquad (3.3)$$

where

$$Eb(\lambda,\mu) = G_0^0(\lambda,\mu) + (1-\lambda^2)\left(-\frac{5}{2}\,G_0^0(\lambda,\mu)\right) + (1-\lambda^2)^2\left(\frac{5}{4}\,G_1^0(\lambda,\mu) - \frac{5}{8}\,G_0^0(\lambda,\mu)\right) + \\ + (1-\lambda^2)^3\left(-\,G_3^0(\lambda,\mu) - 2\,G_2^0(\lambda,\mu) - \frac{37}{24}\,G_1^0(\lambda,\mu) + \frac{11}{6}\,G_0^0(\lambda,\mu)\right) \tag{3.4}$$

$$Db(\lambda,\mu) = (1-\lambda^2)\left(545\,D_1^0(\lambda,\mu) - 265\,D_0^0(\lambda,\mu)\right) + \\ + (1-\lambda^2)^2\left(-\,30\,D_3^0(\lambda,\mu) - 1115\,D_2^0(\lambda,\mu) - \frac{405}{4}\,D_1^0(\lambda,\mu) + \frac{3825}{8}\,D_0^0(\lambda,\mu)\right) + \\ + (1-\lambda^2)^3\left(80\,D_5^0(\lambda,\mu) + 40\,D_4^0(\lambda,\mu) + 580\,D_3^0(\lambda,\mu) + \right. \\ \left. + \frac{345}{2}\,D_2^0(\lambda,\mu) - \frac{2705}{8}\,D_1^0(\lambda,\mu) - \frac{215}{2}\,D_0^0(\lambda,\mu)\right). \tag{3.5}$$

In analogy with (2.54) we can express the density in a form suitable for practical calculation:

$$\rho_c(r,\boldsymbol{n}) = \frac{1}{4\pi\,\kappa\,R}\left(\frac{15}{2}\,(1-s^2)\,g(R,\boldsymbol{n}) - \right. \\ - \frac{1}{2}\int_{-1}^{1} \mathrm{d}\sigma\;\Delta db_1(s,\sigma)\;e(s,\sigma)\;\Sigma_1(\boldsymbol{n}, b(s,\sigma))\,g(R,*) - \\ \left. - \frac{1}{2}\int_{-1}^{1} \mathrm{d}\tau\;\Psi db_1(s,\tau)\;h(s,\tau)\;\Sigma_1(\boldsymbol{n}, a(s,\tau))\,g(R,*)\right) + \\ + \frac{1}{2}\int_{-1}^{1} \mathrm{d}\tau\;\Gamma eb_0(s,\tau)\;\Sigma(\boldsymbol{n}, a(s,\tau))\,\rho(R,*), \tag{3.6}$$

where

$$\Delta db_1(\lambda,\sigma) = \frac{1}{2}\,Pdb_0(1-\lambda^2)\;\Delta(\lambda,\sigma)\;(1-\lambda\sigma)^2 + Pdb_1(1-\lambda^2) \tag{3.7}$$

$$\Psi db_1(\lambda,\tau) = \textstyle\sum_{2\le l\le 5} Pdb_l(1-\lambda^2)\;(1-\lambda^2)^{3-2l}\;(1+\lambda\tau)^{2(l-2)} \tag{3.8}$$

$$\Gamma eb_0(\lambda,\tau) = \textstyle\sum_{0\le l\le 3} Peb_l(1-\lambda^2)\;(1-\lambda^2)^{-2l-1}\;(1+\lambda\tau)^{2l} \tag{3.9}$$

and

$$Pdb_0(u) = -\,u\,(1075 - 1340\,u) \\ Pdb_1(u) = \frac{1}{12}\,u^2\,(20220 - 20455\,u + 3415\,u^2) \\ Pdb_2(u) = \frac{1}{8}\,u^3\,(1512 - 17484\,u + 9707\,u^2) \\ Pdb_3(u) = \frac{1}{4}\,u^4\,(600 - 2115\,u + 2860\,u^2) \\ Pdb_4(u) = -\,\frac{1}{2}\,u^6\,(350 - 385\,u) \\ Pdb_5(u) = \frac{315}{8}\,u^8 \tag{3.10}$$

$$Peb_0(u) = \frac{1}{24}\, u\,(24 - 120\,u + 161\,u^2 - 65\,u^3)$$

$$Peb_1(u) = -\,\frac{1}{16}\, u^3\,(48 - 294\,u + 205\,u^2)$$

$$Peb_2(u) = \frac{15}{8}\, u^5\,(4 - 5\,u)$$

$$Peb_3(u) = -\,\frac{35}{16}\, u^7. \tag{3.11}$$

It has to be emphasized that, in view of (3.3), the characteristic density is a linear integral transformation of the input and, therefore, it can be calculated for any form of the input (if the integration can be performed); the only connection to the above-mentioned model situation is represented by the form of integral kernel $Db(s, \boldsymbol{n}{\cdot}\boldsymbol{n}')$. More exactly, the form of this kernel is such that for function $g(R, \boldsymbol{n})$ representing the gravity field of a point source located within the body, the second term on the r.h.s. of (3.3) is a function with a local extremum at this point (this function can be called the elementary density corresponding to this point source). As the external gravity field of the body is in any case a linear superposition of the fields of (a finite or infinite number of) point sources located within the body, the second term on the r.h.s. of (3.3) is a linear superposition of the corresponding elementary densities.

As the elementary density is a relatively complicated function, it is not possible to discuss its properties here in detail; even in IPG this has only been done in the case in which depth d of the point source is much smaller than the radius of the body (this corresponds to the approximation of the surface of the body by a plane). We can mention only the most important properties of this function: it depends linearly on the mass m of the point source; it is equal to zero at the surface of the body; it has its main extremum at the point source and its value at this extremum is (in this approximation) $5m/2\pi d^3$; there is another extremum on the circle lying at the same depth as the source (with its centre at the point source), the radius of this circle is $\sim 1.343\,d$ and the value at this extremum is ~ -0.113 of the value at the main extremum. It may seem strange that the elementary density is not of the same sign everywhere, but it can be shown that this is the price paid for the linear dependence of this density on the input.

Nevertheless, the fact that the characteristic solution is a linear integral transformation of the input represents the main advantage of the presented method, as this solution is easily to handle numerically: it is not necessary to use any iterative methods that usually require much computation time.

Finally, it should be noted that the choice of the particular form of the characteristic solution does not prevent other solutions of the inverse problem from being considered: any solution can be obtained from the characteristic solution by adding a density of form (2.39) or (2.55) with a suitably chosen function $Z(r, \boldsymbol{n})$. Moreover, the detailed calculation of the characteristic solution in IPG allows other of its forms to be found if some other criteria for its determination are imposed.

4. References

Pohánka V., 1993: *Inverse problem of gravimetry for a spherical planetary body.*
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