Maximally regular net for the rotational ellipsoid

V. Pohánka
Geophysical Institute of the Slovak Academy of Sciences[1]

Abstract: The maximally regular net on the unit sphere is adapted for the surface of the rotational ellipsoid. The mapping between these two surfaces is performed by using the ellipsoidal coordinates. The values of coordinates of vertices of the net at the surface of the Earth reference ellipsoid WGS 84 were calculated from the coordinates of vertices at the unit sphere.

Key words: ellipsoidal surface, mapping, vertex

1. Introduction

The maximally regular net of domains partitioning the surface of the unit sphere was defined in the author's work *Pohánka (2006)* (in the sequel referred to as NS). As it was already stated there (Section 8), this net can be extended to any smooth surface, which does not differ too much from the spherical surface. Among such surfaces the most important case is the surface of the rotational ellipsoid, as many planetary bodies have approximately this shape. The aim of this work is to describe the mapping of the domains of the net from the unit sphere to the rotational ellipsoid. This mapping uses the ellipsoidal coordinate system and it represents a natural adaptation of the net to the ellipsoidal surface. However, it does not necessarily mean that the resulting net is the maximally regular net at the surface of the rotational ellipsoid (it is questionable whether there is a reasonable criterion for the distinguishing of such a net).

[1] Dúbravská cesta 9, 845 28 Bratislava, Slovak Republic
e-mail: Vladimir.Pohanka@savba.sk

2. Adaptation of the net for the rotational ellipsoid

Let us have a body of the shape of rotational ellipsoid whose equatorial radius is a and the excentricity is ε, where $0 \le \varepsilon < 1$. If we introduce the ellipsoidal coordinates (coordinates of oblate spheroid) υ, ξ, ψ, where $\upsilon \ge 0$, $0 \le \xi \le \pi$, $0 \le \psi < 2\pi$, we can write the radius-vector of arbitrary point in the form

$$\boldsymbol{r}(\upsilon, \xi, \psi) = a\,(p(\upsilon) \sin\xi \cos\psi,\ p(\upsilon) \sin\xi \sin\psi,\ q(\upsilon) \cos\xi), \tag{1}$$

where

$$p(\upsilon) = \sqrt{(1-\varepsilon^2)\upsilon^2 + \varepsilon^2}, \quad q(\upsilon) = \sqrt{1-\varepsilon^2}\,\upsilon \tag{2}$$

(see *Bateman* and *Erdélyi, 1953*, 16.1.3, and *Pohánka (1995)*, Section 6). The surface defined by the constant value of the coordinate υ is (for $\upsilon > 0$) an ellipsoidal surface with the excentricity $\varepsilon / p(\upsilon)$; for $\upsilon = 0$ it degenerates to a disk with radius $a\varepsilon$. The surface of the body is defined by the condition $\upsilon = 1$, while in the interior of the body we have $0 \le \upsilon < 1$.

The distance of a point with the radius-vector $\boldsymbol{r}(\upsilon, \xi, \psi)$ from the centre of the body is given by the expression

$$|\boldsymbol{r}(\upsilon, \xi, \psi)| = a\sqrt{(1-\varepsilon^2)\upsilon^2 + \varepsilon^2 \sin^2\!\xi}. \tag{3}$$

The unit vector $\boldsymbol{n}(\upsilon, \xi, \psi)$ of the external normal to the surface defined by the constant value of υ is proportional to $\partial_\xi \boldsymbol{r}(\upsilon, \xi, \psi) \times \partial_\psi \boldsymbol{r}(\upsilon, \xi, \psi)$, and thus it is equal to

$$\boldsymbol{n}(\upsilon, \xi, \psi) = \frac{1}{k(\upsilon, \xi)}\,(q(\upsilon) \sin\xi \cos\psi,\ q(\upsilon) \sin\xi \sin\psi,\ p(\upsilon) \cos\xi), \tag{4}$$

where

$$k(\upsilon, \xi) = \sqrt{(1-\varepsilon^2)\upsilon^2 + \varepsilon^2 \cos^2\!\xi}. \tag{5}$$

Finally, it is useful to introduce the unit vector

$$\boldsymbol{u}(\xi, \psi) = (\sin\xi \cos\psi,\ \sin\xi \sin\psi,\ \cos\xi); \tag{6}$$

note that it is equal to the limit of the vector $\boldsymbol{r}(\upsilon,\xi,\psi)/a\,p(\upsilon)$ for $\upsilon \to \infty$.

Now let us consider the (abstract) unit sphere used to define the domains of the net and for any point P of this sphere let $\boldsymbol{v}(P)$ be its radius-vector (see NS, Sections 2, 3 and 4). In the case of the spherical surface, the normal vector to the surface at each point of this surface is proportional to the radius-vector of this point of the surface. However, in the case of the rotational ellipsoid, these two vectors have in general not the same direction, and therefore we have several alternatives for the definition of the mapping of points of the unit sphere to the surface of the rotational ellipsoid.

Thus, let P_{e} be the point at the surface of the body (whose ellipsoidal coordinates are $(1,\xi,\psi)$) which should correspond to the point P. We could identify the vector $\boldsymbol{v}(P)$ with the unit vector $\boldsymbol{r}(1,\xi,\psi)/|\boldsymbol{r}(1,\xi,\psi)|$ (proportional to the radius-vector of the point P_{e}) or with the unit vector $\boldsymbol{n}(1,\xi,\psi)$ (normal to the surface of the body at the point P_{e}). However, we do not accept any of these two alternatives, but we identify the vector $\boldsymbol{v}(P)$ with the unit vector $\boldsymbol{u}(\xi,\psi)$.

The reason for this choice is that it allows to extend easily the definition of the point P_{e} for any value of the coordinate υ. The identification $\boldsymbol{v}(P)=\boldsymbol{u}(\xi,\psi)$ defines the mapping which for any point P of the unit sphere coordinates the point $P_{\mathrm{e}}(\upsilon)$ with ellipsoidal coordinates (υ,ξ,ψ) and radius-vector given by (1) (of course, it holds $P_{\mathrm{e}}(1)\equiv P_{\mathrm{e}}$). Then we can construct (uniformly with respect to υ) the division of any ellipsoidal surface defined by the constant value of υ. Thus we introduce for any vertex $V(\mathrm{T})$ at the unit sphere (see NS, Section 5) the corresponding vertex $V_{\mathrm{e}}(\mathrm{T})$ at the surface of the rotational ellipsoid and the vertex $V_{\mathrm{e}}(\upsilon,\mathrm{T})$ at the surface defined by the constant value of υ. Similarly, we introduce for any domain $S(\Sigma)$ at the unit sphere (see NS, Sections 3 and 4) the corresponding domain $S_{\mathrm{e}}(\Sigma)$ at the surface of the rotational ellipsoid and the domain $S_{\mathrm{e}}(\upsilon,\Sigma)$ at the surface defined by the constant value of υ.

Of course, the adopted mapping to the ellipsoidal surfaces results in a distortion of the size and shape of the domains of the net (it can be shown that this distortion is in our case smaller than by the other two alternatives mentioned above). The measure of this distortion is the square of the excentricity; therefore it is reasonable to restrict ourselves to the surfaces defined by the condition $\upsilon \ge \upsilon_0$, where $\varepsilon^2 \ll \upsilon_0^2 \le 1$ (provided $\varepsilon^2 \ll 1$).

The coordinate υ corresponds in the ellipsoidal coordinate system to the radial spherical coordinate: if we define some regular division of the interval $\upsilon_0 \le \upsilon \le 1$, we can construct a system of 3-dimensional domains dividing regularly the upper layer of the body (from the surface $\upsilon = \upsilon_0$ to the surface of the body).

Moreover, our choice has another advantageous aspect: let us consider for each υ such that $0 < \upsilon \le 1$ an ellipsoidal body whose surface is defined by the particular value of υ; let the body have a constant density such that the total mass of each body is equal to some given value. Then each such body has the same external gravity field at any point $\boldsymbol{r}(\upsilon, \xi, \psi)$ with $\upsilon \ge 1$. This means that the surfaces defined by the constant value of υ represent the generalization of the concentric spherical surfaces in the case of spherical body.

For the convenience of the reader we present here the formulae allowing to calculate the vector $\boldsymbol{u}(\xi, \psi)$ from the known position of the point $P_{\mathrm{e}}(\upsilon)$. This position can be given either by the normal vector $\boldsymbol{n}(\upsilon, \xi, \psi)$ and the radial coordinate υ (if we know the direction of the local vertical and the ellipsoidal height) or directly by (the components of) the radius-vector $\boldsymbol{r}(\upsilon, \xi, \psi)$ (if we know the rectangular geocentric coordinates of the point). Let $\boldsymbol{\pi}$ be the north pole unit vector

$$\boldsymbol{\pi} = (0, 0, 1); \tag{7}$$

in the case we know $\boldsymbol{n}(\upsilon, \xi, \psi)$ and υ, we get from (4), (2) and (5)

$$\cos\xi = \frac{q(\upsilon)\ \boldsymbol{\pi}\cdot\boldsymbol{n}(\upsilon, \xi, \psi)}{\sqrt{p(\upsilon)^2 - \varepsilon^2(\boldsymbol{\pi}\cdot\boldsymbol{n}(\upsilon, \xi, \psi))^2}},$$

$$k(\upsilon, \xi) = \frac{p(\upsilon)\, q(\upsilon)}{\sqrt{p(\upsilon)^2 - \varepsilon^2(\boldsymbol{\pi}\cdot\boldsymbol{n}(\upsilon, \xi, \psi))^2}}$$

and finally

$$\boldsymbol{u}(\xi, \psi) = \frac{p(\upsilon)\, \boldsymbol{n}(\upsilon, \xi, \psi) - (p(\upsilon) - q(\upsilon))\, (\boldsymbol{\pi}\cdot\boldsymbol{n}(\upsilon, \xi, \psi))\, \boldsymbol{\pi}}{\sqrt{p(\upsilon)^2 - \varepsilon^2(\boldsymbol{\pi}\cdot\boldsymbol{n}(\upsilon, \xi, \psi))^2}}. \tag{8}$$

In the case we know $\boldsymbol{r}(\upsilon, \xi, \psi)$, we get from (1), (2) and (3)

$$q(\upsilon)^2 = \frac{1}{2\,a^2}\Big(\boldsymbol{r}(\upsilon, \xi, \psi)^2 - a^2\varepsilon^2 + \\ + \sqrt{(\boldsymbol{r}(\upsilon, \xi, \psi)^2 - a^2\varepsilon^2)^2 + 4\,a^2\varepsilon^2(\boldsymbol{\pi}\cdot\boldsymbol{r}(\upsilon, \xi, \psi))^2}\Big), \tag{9}$$

thus

$$\upsilon = \frac{q(\upsilon)}{\sqrt{1-\varepsilon^2}}, \quad p(\upsilon) = \sqrt{q(\upsilon)^2 + \varepsilon^2} \tag{10}$$

and finally

$$\boldsymbol{u}(\xi,\psi) = \frac{q(\upsilon)\,\boldsymbol{r}(\upsilon,\xi,\psi) + (p(\upsilon)-q(\upsilon))(\boldsymbol{\pi}\cdot\boldsymbol{r}(\upsilon,\xi,\psi))\,\boldsymbol{\pi}}{a\,p(\upsilon)\,q(\upsilon)}. \tag{11}$$

3. The net at the surface of the Earth reference ellipsoid

The calculation of rectangular coordinates of vertices of the net at the surface of the rotational ellipsoid from their values at the surface of the unit sphere is a simple operation: from the equations (1) and (2) we easily obtain that for $\upsilon = 1$

$$\boldsymbol{r}(1,\xi,\psi) = (a\,\sin\xi\cos\psi,\ a\,\sin\xi\sin\psi,\ b\,\cos\xi), \tag{12}$$

where b is the polar radius of ellipsoid

$$b = \sqrt{1-\varepsilon^2}\,a. \tag{13}$$

Comparing the formula (12) with (6) we see that the mapping from the surface of the unit sphere to the surface of the rotational ellipsoid with radii a and b consists simply in multiplying the x and y coordinates of each point of the unit sphere by a and the z coordinate by b.

The numerical values of rectangular coordinates of vertices of the net at the surface of the unit sphere were calculated for degrees between 0 and 14 (see *Pohánka (2007)*, in the sequel referred to as NSC). Recall that the net of degree n contains $10 \cdot 4^n + 2$ vertices, thus for $n = 14$ this amounts to 2 684 354 562 vertices. As it was described in detail in NSC, Section 2, the calculation was performed in the extended `long double` representation of real numbers which allows to store 66 significant bits of every number whose absolute value is less than or equal to 2 (compared with 64 bits in the standard `long double` representation and 53 bits in the `double` representation).

The parameters of the Earth reference ellipsoid were taken from the WGS 84 system where the equatorial radius is $a = 6378137\,\mathrm{m}$ and the flattening is $f = 1.0/298.257223563$ (note that $f = (a - b)/a$). The actually used values of a and b (calculated from a and f) were a little different: the reason is that their values are stored in the binary format and thus we have chosen the actual values as binary numbers maximally close to the given ones with 10 significant hexadecimal digits. Moreover, in order to apply the arithmetics developed for the extended `long double` representation of real numbers, we used in the calculation the values of a and b multiplied by 2^{-23} (these values are then smaller than 1). The adopted values of these constants are (in the hexadecimal representation)

$$a \cdot 2^{-23} = \texttt{0XC.2A5320000P} - 4, \quad \texttt{b} \cdot 2^{-23} = \texttt{0XC.1FE20A0E5P} - 4, \tag{14}$$

and thus

$$a = 6378137.000000000000, \quad b = 6356752.314247131348, \tag{15}$$

what gives $f = 1.0/298.257223590223$.

The calculation of the rectangular coordinates of vertices of the net at the surface of the Earth reference ellipsoid was performed in the extended `long double` representation (in order to achieve the maximal possible accuracy); the data are stored in this representation (36 bytes for each vertex) and also in the `double` representation (24 bytes for each vertex). The way of storing the data is the same as in the case of the unit sphere (for the detailed description see NSC, Section 3). We present here for brevity only the basic facts about the data files for net with degrees $12 \le n \le 14$: for the degree $n = 12$ there are 10 data files, for $n = 13$ there are 40 data files, and for $n = 14$ there are 160 data files; each data file contains 16 785 409 vertices and it occupies 604 274 724 bytes (in the extended `long double` representation) or 402 849 816 bytes (in the `double` representation).

4. The orientation of the net at the surface of Earth

The calculated coordinates of vertices of the net are rectangular coordinates where the positive z-axis points towards the north pole of the Earth; the positive x-axis then points to some point B_e at the equator of the Earth.

Let λ_0 be the geographic longitude of the point $B_{\rm e}$; then the correspondence between the ellipsoidal coordinates ξ, ψ and the geographic coordinates ϕ, λ is given by the equation

$$\boldsymbol{n}(1,\xi,\psi)=\boldsymbol{u}(\pi/2-\phi,\lambda-\lambda_0); \tag{16}$$

according to (4) and (6) this implies (modulo 2π)

$$\psi=\lambda-\lambda_0. \tag{17}$$

The value of λ_0 can be chosen (in principle) arbitrarily; in the case of the usual geographic coordinates the point $B_{\rm e}$ is the intersection of the equator with the prime (Greenwich) meridian (what corresponds to $\lambda_0=0$), but, from reasons given below we prefer another choice.

Consider the simplest case, the net of degree 0 at the unit sphere (with 12 vertices, 20 domains and 30 edges): the only edges of this net identical with a part of some meridian are those connecting the north pole $V(00)$ with the vertices $V(a0)$ of the northern ring and those connecting the south pole $V(01)$ with the vertices $V(a1)$ of the southern ring (where a is a digit from among $\{1,2,3,4,5\}$; see NS, Section 2). According to the formulae (3), (14), (15) of NS, within each ring the neighbouring vertices are 72 degrees in longitude apart and the two rings are mutually displaced by 36 degrees in longitude. After projecting the vertices $V(ap)$ of the net of degree 0 from the unit sphere to the rotational ellipsoid using the formulae (6) and (12), we can conclude that the same differences in longitude apply for the vertices $V_{\rm e}(ap)$ at the surface of the Earth reference ellipsoid.

Therefore, it has no sense to fix some vertex at the prime meridian. Instead of this, we find it advantageous to fix the boundaries of domains in such a way that, if possible, the well separated continents are also separated by boundaries of domains. As we have only a single free parameter, we propose that Asia and North America should be separated by an edge connecting the north pole with some vertex of the northern ring. On the other hand, we require that the value of λ_0 (expressed in degrees) should be a simple number (more exactly, it should be a simple fraction of the full circle). This can be at best achieved if we choose this value as -24 degrees. The geographic longitudes of all vertices of the northern and southern ring (in degrees) are then given in the Table 1. The geographic latitude of vertices of both rings can be easily calculated using the formulae (14), (15) of

NS, and (4), (5) and (6) from the present paper: the latitude of vertices of the northern ring is 26.642098551 degrees (and opposite for the vertices of the southern ring).

Table 1. Geographic longitudes of the vertices of the northern and southern ring

$V_e(10)$	$V_e(20)$	$V_e(30)$	$V_e(40)$	$V_e(50)$
−24	48	120	−168	−96

$V_e(11)$	$V_e(21)$	$V_e(31)$	$V_e(41)$	$V_e(51)$
12	84	156	−132	−60

5. Final note

The data files containing the calculated coordinates of vertices of the net at the surface of the Earth reference ellipsoid can be obtained from the author by a personal request.

Acknowledgments. The author is grateful to VEGA, the Slovak Grant agency (projects No. 2/6019/26 and 1/3066/06), and to the Science and Technology Assistance Agency of the Slovak Republic (contract No. APVV-99-002905) for the partial support of this work.

References

Bateman H., Erdélyi A., 1953: Higher Transcendental Functions. New York, McGraw-Hill.

Pohánka V., 1995: Solution of the Dirichlet and Neumann boundary problems for the Laplace equation for the rotational ellipsoid by the integral equation method. Contr. Geophys. Geod., **25**, 55–72.

Pohánka V., 2006: The maximally regular net on the sphere. Contr. Geophys. Geod., **36**, 115–152.

Pohánka V., 2007: Calculation of the maximally regular net on the sphere. Contr. Geophys. Geod., **37**, 359–376.