Calculation of the maximally regular net on the sphere

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A b s t r a c t : The method used for the numerical calculation of the maximally regular net on the sphere is described and the results of the calculation are presented. The calculation required to develop the fixed point representation of real numbers and procedures for manipulation with such data. The results include several empirical formulae for the number of occurences of values of coordinates of vertices of the net and of values of scalar product of radius-vectors of neighbouring vertices, whose theoretical derivation seems to be very difficult.

Key words: spherical triangle, vertex, accuracy, fixed point representation

1. Introduction

The aim of this paper is to describe the numerical calculation of the rectangular coordinates of vertices of the maximally regular net on the sphere presented by *Pohánka (2006)* (in the sequel referred to as NS). We recall that this net of triangular domains (and their vertices) is defined for each natural number n (called the degree of the net) in a very simple way as follows. The domains of the net of degree 0 are central projections of the sides of the regular icosahedron on the unit sphere (see *Mathworld*). For any n (where $n \geq 0$), each triangular domain of degree n is divided into four smaller triangular domains by joining the centres of edges of the original domain by segments of great circles (the smaller domains have then the degree $n+1$).

Despite this simplicity, the actual numerical calculation of the coordinates of vertices of these domains is not so straightforward, if we want to achieve the maximal numerical accuracy. As the coordinates of vertices are

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calculated iteratively, it is evident that the accuracy of the numerical values of coordinates will decrease with the increasing degree. Therefore it would be very advantageous to calculate the coordinates of the vertices of certain (suitably chosen) degree as accurately as possible. Then the coordinates of vertices of any larger degree can be obtained from those already calculated with greater accuracy than by their direct calculation from the net of degree 0.

2. Method of calculation

The calculation of the coordinates of vertices was performed for the net of degree $n = 14$. The reason for this choice was twofold: first, this value means that each vertex of the net will have the code containing 16 digits. As explained in NS, Section 5, the code of each vertex of degree n is a sequence of $n+2$ digits; for the vertices which are not the polar ones, the first digit is from among $\{1, 2, 3, 4, 5\}$, the second one is from among $\{0, 1\}$ and the other ones are from among $\{0, 1, 2, 3\}$ (for the two polar vertices, the second digit is as before, while all other digits are equal to 0).

The second reason for the choice of this degree is that in the case of the sphere with the radius equal to the earth equatorial radius, the elementary edges of the net have the length of approximately 0.43 km (the length of the edges is variable but confined in a narrow interval). Thus this net is sufficiently dense for the practical purposes. Moreover, the still greater value of degree would require enormous storage space: the net of degree n contains $10 \cdot 4^n + 2$ vertices, thus for $n = 14$ this amounts to 2 684 354 562 vertices.

In order to calculate the coordinates of these vertices with the maximal possible accuracy, we have chosen to use the `long double` representation of real numbers. The precision of any representation of real numbers can be expressed by the value of the minimal difference of the representation which can be defined as follows.

The minimal difference δ of the representation is the difference of any two nearest different real numbers from the interval $\langle 0.5, 1.0\rangle$ expressible in this representation. It has to be noted that for the other choice of the interval the difference of the two nearest real numbers is different: for example, for

the interval $\langle 0.25, 0.5\rangle$ it is $\delta/2$, while for the interval $\langle 1.0, 2.0\rangle$ it is 2δ. As the characterization of the accuracy, the minimal difference is better than the relative accuracy, which is not a constant: it lies in the interval $(\delta, 2\delta)$.

The minimal difference of the `long double` representation is $\delta_{\mathrm{ld}} = 2^{-64}$, while for the `double` representation it is $\delta_{\mathrm{d}} = 2^{-53}$. In other words, in the former case the mantissa of the representation has 64 significant bits, while in the latter case only 53 bits.

The choice of the most accurate available representation is only the first step in the calculation of the net; the second one is the use of the appropriate arithmetics of calculation. The floating point arithmetics used in computers is not the best one for the case of simple operations with the coordinates of points on (or in) the unit sphere. The reason is in the uniformity of the representations of real numbers with respect to the multiplication: this means that the relative accuracy of the product of two numbers is of the same order as the relative accuracy of these numbers. On the other hand, the relative accuracy of the sum (or difference) of two numbers can be much smaller than the relative accuracy of these numbers. Further, the absolute accuracy of the sum of two numbers can be smaller (up to twice) than the absolute accuracy of these numbers. Another unpleasant effect is that the result of summation of three (or more) numbers may depend on the order of summation; in order to obtain a unique result it is necessary to order the numbers according to their absolute value and to add them in the order from the smallest to the largest one.

The floating point representation may also hinder the comparison of two real numbers with close values. In our case it is necessary to test the absolute value of the (calculated) unit vector with respect to the expected value 1. This is complicated by the fact that the nearest number smaller than 1.0 expressible in the floating point representation is $1.0 - \delta$, while the nearest number greater than 1.0 is $1.0 + 2\delta$. Thus there is no uniform measure of the deviation of a number from the value 1.0.

In the calculation of the net we have to perform mostly the operations of the summation and scalar multiplication of two unit vectors (note that by the latter operation we calculate the sum of three products of components of these vectors). Thus the negative properties of the floating point arithmetics apply fully to our case.

Let us consider the radius-vectors of the points on or in the unit sphere:

any component of any such vector is absolutely smaller than or equal to 1. In order to represent the position of these points with an uniform accuracy we need to know the components of their radius-vectors with some given absolute accuracy (the relative accuracy of these components is irrelevant). Then the sum of two (or more) vectors will be the vector whose components will have the same absolute accuracy. Similarly, the scalar product of two such vectors will have also this absolute accuracy (as this product is always absolutely smaller than or equal to 1).

In order to conform to these requirements, it is necessary to adopt the suitable fixed point representation of real numbers and the suitable fixed point arithmetics (uniform with respect to the addition). The representation and arithmetics used in our calculation are described below.

The adopted fixed point representation of real numbers has two forms: one is used in the arithmetical operations with these numbers, while the other is used for the storage of these numbers. We begin with the former (called the internal representation): the real number x is represented as the pair (k, q), where k is a `long int` number and q is a `long double` number such that $0.0 \leq |q| < 0.25$ and the sign of k (if it is nonzero) is the same as the sign of q; the value of the real number represented by this pair is $x = 0.25k + q$ (it is clear that this representation is unique). Moreover, the value of q is normalized: this means that for any q such that $0.0 \leq |q| < 0.125$ the value of q is rounded to the absolute accuracy 2^{-66} (this is the accuracy of any q such that $0.125 \leq |q| < 0.25$). The pair (k, q) is declared as the `struct lde4` number (it means the `long double` extended by the factor 4). The minimal difference of this representation is thus $\delta_{\mathrm{lde4}} = 2^{-66}$.

For the storing of `struct lde4` numbers we have used the fact that the `long double` number occupies in the memory 12 bytes, but from these only the first 10 bytes contain the value of this number, while the last 2 bytes are arbitrary. These 2 bytes can be used for the storing of additional information; in our case the first byte is zero, while the second byte contains the additional bits of the mantissa of the stored number (the first 64 bits are stored in the first 10 bytes). Thus the second byte can contain from 0 to 4 nonzero bits according to the absolute value of the stored real number x: 0 bits for $|x| < 0.25$, 1 bit for $0.25 \leq |x| < 0.5$, 2 bits for $0.5 \leq |x| < 1.0$, 3 bits for $1.0 \leq |x| < 2.0$, and 4 bits for $2.0 \leq |x| < 4.0$. This is our external

representation of real numbers; it can represent any number x such that $0.0 \leq |x| \leq 4.0$ with the absolute accuracy of 2^{-66} (numbers with larger absolute value do not appear in our calculation).

The adopted fixed point arithmetics is based on the internal representation of real numbers and the usual `long double` arithmetics; it is applicable to real numbers such that $0.0 \leq |x| \leq 4.0$. It contains several procedures for the manipulation with the `struct lde4` numbers: procedure `sume4` for the summation of two numbers, `multe4` for the multiplication of two numbers, `invep4` for the inverse value of the positive number, `sqrtep4` for the square root of the positive number, and other ones. There are also procedures `ldtoe4` and `e4told` for the conversion of the external to the internal representation and vice versa, and the procedure `printe4` for the text output of `struct lde4` numbers.

For the operations with vectors there is introduced the structure `struct vecte4` containing three `struct lde4` vector components. The procedure `vecte4prod` calculates the scalar product of two vectors; as all `struct lde4` numbers are normalized, the value of the scalar product does not depend on the order of summation.

Now we can describe the calculation of the coordinates of the vertices of the net. As explained in NS, Sections 2 and 5, each vertex is denoted by the sequence of digits T and expressed by the unit radius-vector $\boldsymbol{e}(\mathrm{T})$. The net of degree 0 contains two polar vertices with codes 00 and 01

$$\boldsymbol{e}(00) = (0,\, 0,\, 1), \quad \boldsymbol{e}(01) = (0,\, 0,\, -1), \tag{1}$$

and 10 vertices (northern ring vertices $a0$ and southern ring vertices $a1$)

$$\boldsymbol{e}(a0) = \frac{1}{\sqrt{5}}(2\,\cos{(2a-2)\psi_5},\, 2\,\sin{(2a-2)\psi_5},\, 1), \tag{2}$$

$$\boldsymbol{e}(a1) = \frac{1}{\sqrt{5}}(2\,\cos{(2a-1)\psi_5},\, 2\,\sin{(2a-1)\psi_5},\, -1), \tag{3}$$

where $a \in \{1, 2, 3, 4, 5\}$ and $\psi_5 = \pi/5$. Using the formulae

$$\cos\psi_5 = \frac{\sqrt{5}+1}{4}, \quad \cos 2\psi_5 = \frac{\sqrt{5}-1}{4}, \tag{4}$$

and $\cos 3\psi_5 = -\cos 2\psi_5$, $\cos 4\psi_5 = -\cos \psi_5$, we can easily calculate

$$\begin{aligned}
&\boldsymbol{e}(10) = (c_6,\, 0,\, c_2), && \boldsymbol{e}(11) = (c_4,\, c_3,\, -c_2),\\
&\boldsymbol{e}(20) = (c_1,\, c_5,\, c_2), && \boldsymbol{e}(21) = (-c_1,\, c_5,\, -c_2),\\
&\boldsymbol{e}(30) = (-c_4,\, c_3,\, c_2), && \boldsymbol{e}(31) = (-c_6,\, 0,\, -c_2),\\
&\boldsymbol{e}(40) = (-c_4,\, -c_3,\, c_2), && \boldsymbol{e}(41) = (-c_1,\, -c_5,\, -c_2),\\
&\boldsymbol{e}(50) = (c_1,\, -c_5,\, c_2), && \boldsymbol{e}(51) = (c_4,\, -c_3,\, -c_2),
\end{aligned} \tag{5}$$

where the constants c_i are defined as follows:

$$\begin{aligned}
&c_1 = \frac{1}{2}(1 - c_2), \quad c_2 = \frac{1}{\sqrt{5}}, \quad c_3 = \sqrt{c_1},\\
&c_4 = \frac{1}{2}(1 + c_2), \quad c_5 = \sqrt{c_4}, \quad c_6 = 2c_2.
\end{aligned} \tag{6}$$

Note that the expression (5), (6) of the radius-vectors is more preferable than the expression (2), (3), as by the numerical calculation of the latter one there can be more mutually different values of components of vectors than by the former one. Nevertheless, even the values of constants c_i calculated according to (6) with the fixed point arithmetics are not their definitive values. The main criterion for their values is that the 12 vectors (1), (5) have to be unit vectors and the scalar product of any pair of neighbouring vectors has to be equal to c_2. Therefore the following test was performed: the scalar product of each pair from among the 12 vectors was calculated for variable values of all 6 constants c_i (differing from the values (6) by a small multiple of δ_{lde4}) and the number of mutually different values of the scalar product was obtained. In the ideal case this number would be 4 (the exact values of the scalar product are in the increasing order -1, $-c_2$, c_2 and 1). The result of the test was that there was the single minimum of different values of the scalar product with 2 values corresponding to 1 and 4 values corresponding to c_2; this minimum was obtained for corrected values of c_1 and c_5. The definitive values of the constants c_i are given below:

$$\begin{aligned}
&c_1 = 0.27639320225002103035, && c_2 = 0.44721359549995793928,\\
&c_3 = 0.52573111211913360602, && c_4 = 0.72360679774997896964,\\
&c_5 = 0.85065080835203993219, && c_6 = 0.89442719099991587857.
\end{aligned} \tag{7}$$

After obtaining the positions of the vertices of degree 0 the calculation of the positions of vertices of degree greater than 0 was possible. As explained

in NS, Section 4, any edge of the net of degree n is divided into two edges of degree n+1 by its centre (this centre is the vertex of degree n+1). The centre of the edge whose endpoints have radius-vectors $\boldsymbol{u}$, $\boldsymbol{v}$, has the radius-vector $\boldsymbol{c}(\boldsymbol{u}, \boldsymbol{v})$, where

$$\boldsymbol{c}(\boldsymbol{u}, \boldsymbol{v}) = \frac{\boldsymbol{u} + \boldsymbol{v}}{|\boldsymbol{u} + \boldsymbol{v}|}. \tag{8}$$

Thus the calculation of the coordinates of new vertices according to this simple formula seems to be an easy task. Nevertheless, as in the case of degree 0, the numerical calculation may yield vectors whose (squared) absolute value differs from 1 by more than it is desirable. Moreover, there is another circumstance, which is even more important for the accuracy of the calculated net of degree $n > 0$. Consider the set of all values of (calculated) coordinates of vertices of some order; this set contains besides 0 some number of positive and the same number of negative values of coordinates (with respect to the method of their calculation there is for every positive value the single negative value which differs from the former only by its sign). As shown above, the number of positive values of coordinates for the net of degree 0 is 7. By the numerical calculation of the net of degree $n+1$ from the net of degree n it can happen that for each theoretical value from the set of all values there are several calculated numerical values (instead of the expected single one). However, on the contrary to the case of degree 0, it is hardly possible to derive some useful formulae (similar to (6)) for the theoretical values of coordinates of degree $n > 0$.

Therefore, for the recognition of the multiple values (corresponding to the single theoretical value) there was applied the following simple criterion: if the difference of two calculated values of coordinates is smaller than $5\,\delta_{\mathrm{lde4}}$, then these two values belong to the single theoretical value. Of course, if it happens that the two theoretical values are less than $5\,\delta_{\mathrm{lde4}}$ apart, then they are merged into a single value. This case was not expected, as a very crude consideration shows that the positive values of coordinates should be distributed almost uniformly in the interval $\langle 0.0, 1.0\rangle$; thus the difference between two successive values of degree n should be roughly of the order 4^{-n}, what is for $n \leq 14$ much greater than $4\,\delta_{\mathrm{lde4}}$.

The multiple calculated values of coordinates have to be corrected to obtain the single value; otherwise the multiplicity of values would grow

with increasing degree and for the net of degree 14 it could be unacceptably high. The correction was performed successively (as there can be up to five values corresponding to the single theoretical value). For each value there was first obtained the statistics showing for each integer k (where $-4 \leq k \leq 4$) the number of occurences of this value in vectors with squared length of $1.0+k\,\delta_{\mathrm{lde4}}$. This statistics was then used in the process of choosing the resulting single value with the aim to minimize the negative effects of the choice: it is clear that the change in one component of a vector results (mostly) in the change of its squared length.

Despite this aim the process of replacing multiple values of coordinates by a single one increases the number of vectors with squared length differing from 1.0 by more than it is acceptable. For brevity we shall say that the squared length of a vector is admissible just if it is in the interval $\langle 1.0-\delta_{\mathrm{lde4}}, 1.0+\delta_{\mathrm{lde4}}\rangle$; such vector will be called the admissible one. The original vectors (before the described reduction) were mostly admissible and only a few had the squared length differing from 1.0 by $2\,\delta_{\mathrm{lde4}}$. On the contrary, after the reduction there were relatively many vectors with squared length differing from 1.0 by $2\,\delta_{\mathrm{lde4}}$, $3\,\delta_{\mathrm{lde4}}$ or even more.

Therefore, the next step after the reduction of multiple values was the correction of values of components of inadmissible vectors in order to decrease the number of them; the ultimate goal was that every vector would be admissible. This second correction of the values of components of vectors is not an easy task, as the change of a single value results in the change of the squared length of all vectors in which this value appears as a component. Thus making some vectors admissible can make other vectors inadmissible.

After several attempts there was finally developed an algorithm for the iterative correction of the set of values of components of vectors reducing in each iterative step the number of inadmissible vectors. This algorithm can be described shortly as follows: in each iteration there is first found the set of all inadmissible vectors and (as above) the set of all values of components of all vectors together with the statistics showing for each value and for each integer k the number of occurences of this value in vectors with squared length of $1.0+k\,\delta_{\mathrm{lde4}}$. For each inadmissible vector the evaluation of this statistics of its components is performed in order to find the component that is chosen for the correction of its value. Moreover, if this choosing is successful, all components of this vector are marked; if for some

next inadmissible vector there is some value of one of its components that is already marked, this vector is left unchanged. After checking all inadmissible vectors the correction of values is performed in all data files and the next iterative step can follow. The algorithm stops if there is no possibility to change any value.

The result of the application of this algorithm represents the definitive set of vectors of vertices of the given degree; this set can be then used in the calculation of the vectors of vertices of the next higher degree. In this way there were obtained the definitive data files containing for each degree $n \leq 14$ the components of radius-vectors of vertices of this degree.

3. Storing the calculated data

We describe here the way of storing the coordinates of vertices in the data files. As it was mentioned in the previous section, each coordinate is stored in 12 bytes as a `long double` number (where the last two bytes are also necessary for the recovering of the whole `struct lde4` value). Each vertex thus occupies 36 bytes containing its x, y, z coordinates. The order of vertices in the data files is not conforming their denotation, but the natural order, which is as follows. According to NS, Section 3 (and Fig. 1 therein), if we unfold the surface of the regular icosahedron, we obtain the 20 planar triangular domains which can be grouped in 10 pairs of domains (each pair has the common base vertex, see NS, Section 5). The coordinates of vertices are stored in 10 data files each containing the vertices of one pair of triangular domains (thus all vertices in the domains and at their boundary). Therefore the vertices lying at the boundary of each pair of domains are contained in the data files more than once, but this is advantageous by the finding of the neighbours of the given vertex, see NS, Section 7). For each pair of triangular domains, the vertices are stored in rows ordered from the top to the bottom (thus from the north to the south); in each row the vertices are ordered from the left to the right (thus from the west to the east). For the net of degree n, the number of rows is $2^{n+1}+1$; the number of vertices in the row grows linearly from 1 in the first row to 2^n+1 in the middle (thus (2^n+1)-st) row and then decreases linearly to 1 in the last row. The total number of vertices stored in the single data file is thus $(2^n+1)^2$.

The described way of organisation of data files represents the ideal case; the actual organisation is different, because the length of the data file (in the system used by the calculation) cannot be greater or equal than 2 gigabytes. Therefore the described organisation can be used only for the net of degree $n \leq 12$; for larger degree it was necessary to divide the data files into smaller ones. This was done using the natural division of domains described in NS, Section 4 and 5. For the degree $n \leq 12$ the data files (each containing one pair of domains) are denoted by the sequence ap (where $a \in \{1, 2, 3, 4, 5\}$, $p \in \{0, 1\}$) which is the code of the base vertex of this pair of domains. For the larger degree, each pair of triangular domains is divided once (for $n = 13$) or twice (for $n = 14$) into four pairs of triangular domains denoted (in accordance with their base vertex) by the sequence aps_1 (where $s_1 \in \{0, 1, 2, 3\}$) for $n = 13$ and the sequence aps_1s_2 (where $s_2 \in \{0, 1, 2, 3\}$) for $n = 14$. The organisation of the vertices in these divided data files is the same as in the original ones. Thus for the degree $n \leq 12$ there are 10 data files, for $n = 13$ there are 40 data files, and for $n = 14$ there are 160 data files. For the degree $n \geq 12$, each data file contains 16 785 409 vertices and it occupies 604 274 724 bytes.

4. Results

The actual calculation of the data has shown that the procedures for the correction of the values of coordinates described in the Section 2 worked flawlessly with the single exception in the case of degree 14, where it was necessary to perform an additional correction of a single pair of successive values. The occurence of this exceptional case was the consequence of the behaviour of the distribution of values of coordinates of vertices, which was quite different from the expectation. It was confirmed that the positive values of coordinates of vertices are distributed in the interval $\langle 0.0, 1.0\rangle$ almost uniformly, but it turned out that there are several small subintervals, where the values are distributed much densely than in the mean. The minimal difference of the successive values was shown to be greater than $5\delta_{\mathrm{lde4}}$ only for $n \leq 12$, where it decreased with the increasing degree n roughly as 2^{-5n}, thus much faster than expected. For $n > 12$ this decreasing stopped: the minimal difference was $113\,\delta_{\mathrm{lde4}}$ for $n = 12$, and $5\,\delta_{\mathrm{lde4}}$ for

$n = 13$ and $n = 14$. From the behaviour of the minimal difference for $n \leq 12$ it was possible to determine its theoretical value for $n \geq 12$: this value is $113.2\,\delta_{\mathrm{lde4}}$ for $n = 12$, $3.5\,\delta_{\mathrm{lde4}}$ for $n = 13$, and $0.1\,\delta_{\mathrm{lde4}}$ for $n = 14$.

This means that for the degrees 13 and 14 the calculated data contain several values of coordinates which are the result of merging of more theoretical values into a single one. This does not represent any drawback for the use of data in practice, as the merged values can hardly differ from the theoretical value by more than $2\,\delta_{\mathrm{lde4}}$. In this context it is very interesting that for every value in the data it was possible to determine, whether this value is the result of merging of more theoretical values, and, also the number of these merged theoretical values (see below). It turned out that there are values resulting from the merging of 2 theoretical values (totally 1 011 values for $n = 13$ and 2 224 values for $n = 14$) and values resulting from the merging of 4 theoretical values (totally 2 042 values for $n = 14$).

The actual calculation has shown that the procedure of reduction of multiple values of coordinates of vertices worked well for each degree (with the single exception mentioned above) and in each step the multiplicity of values was reduced by 1. For each degree n, the number of positive values with the multiplicity i in the uncorrected data is shown in Table 1 as M_i; the maximal multiplicity was 4. As for the subsequent iterative correction of values of coordinates, in each iterative step the number of inadmissible vectors was reduced by the factor of about 4; the number of iterative steps is shown in Table 1 as I.

For each degree n, the number of resulting positive values, which were in the iterative process corrected i-times, is shown in Table 2 as N_i; the maximal number of corrections was 4. The fact that some value was corrected i-times does not necessarily mean that the total correction was $i\,\delta_{\mathrm{lde4}}$, as for $i \geq 2$ there can be corrections which moved the value in both directions. The total number of resulting positive values is shown in Table 2 as N.

As for the length of the vectors, the resulting data of degree 14 show that there remained only 24 inadmissible vectors, all with the squared length of $1.0 - 2\,\delta_{\mathrm{lde4}}$.

An interesting result of the calculation was the number of occurences of each positive value as (the absolute value of) the coordinate of a vertex in the whole data of the given degree. It could be expected that every positive

Table 1. Number of values of coordinates M_i with mutiplicity $i > 1$ in the uncorrected data and the number of iterations I for each degree n

n	M_2	M_3	M_4	I
0	0	0	0	0
1	2	0	0	1
2	5	0	0	0
3	31	0	0	2
4	72	0	0	2
5	340	1	0	3
6	1385	10	0	4
7	5491	34	0	5
8	21825	183	0	8
9	88428	853	0	7
10	354789	3717	1	8
11	1428180	14955	3	9
12	5713096	62374	10	10
13	22925019	254287	44	12
14	91950261	1035975	175	12

Table 2. Number of values of coordinates N_i corrected i-times and the total number of resulting values N for each degree n

n	N_0	N_1	N_2	N_3	N_4	N
0	7	0	0	0	0	7
1	15	1	0	0	0	16
2	48	1	0	0	0	49
3	249	3	1	0	0	253
4	1053	14	1	0	0	1068
5	4276	53	3	0	0	4332
6	17159	217	11	1	0	17388
7	68551	1022	38	1	0	69612
8	274091	4254	156	7	0	278508
9	1095487	17911	680	14	0	1114092
10	4379348	73947	3068	65	0	4456428
11	17510347	302259	12891	275	0	17825772
12	70026227	1222707	53098	1116	0	71303148
13	280057254	4932473	217120	4790	4	285211641
14	1120057585	19876548	888258	19914	13	1140842318

value appears in the data for the first time for some degree and with the increasing degree the number of occurences of this value will grow (it is clear that this number cannot decrease). However, it turned out that there are only 16 values whose number of occurences has grown; in other words, these values appeared in the new vertices of some degree (these are the vertices which were not present for lower degree) for more than one value of degree. In detail, 3 values appear in new vertices of the degree $n \in \{0, 1, 2\}$, 4 values in the degree $n \in \{0, 2\}$, 8 values in the degree $n \in \{1, 2\}$, and one value in the degree $n \in \{1, 4\}$. All other positive values appear in the new vertices of some degree and then the number of their occurences remains constant with the increasing degree. It turned out that for degrees $2 \leq n \leq 12$ the number of occurences of any positive value in the whole data is always a number from the set $\{2, 4, 6, 8, 10, 12, 14, 20, 24\}$; for degrees $n \geq 13$ there are values whose number of occurences is 16 and for $n = 14$ there are values whose number of occurences is 32.

Moreover, if we define for each degree n the quantity $N(n, o)$ as the number of positive values whose number of occurences is o (where o is a number from the set mentioned above), using the values of $N(n, o)$ for degrees $4 \leq n \leq 12$, we can derive the following empirical formulae for the quantity $N(n, o)$:

$$\begin{aligned}
&N(n,2) = 6 \cdot 2^{n-2} - 5, && N(n,4) = 40 \cdot 4^{n-2} - 5 \cdot 2^{n-2} - 18,\\
&N(n,6) = 2 \cdot 2^{n-2}, && N(n,8) = 20 \cdot 4^{n-2} - 11 \cdot 2^{n-2},\\
&N(n,10) = 6 \cdot 2^{n-2} - 5, && N(n,12) = 5 \cdot 2^{n-2} + 5,\\
&N(n,14) = 2 \cdot 2^{n-2} - 1, && N(n,20) = 8 \cdot 4^{n-2} - 5 \cdot 2^{n-2} + 2,\\
&N(n,24) = 2,
\end{aligned} \tag{9}$$

where $n \geq 4$. There is no reason for these formulae to be not valid for all $n \geq 4$; assuming their validity we obtain from them the expected values of $N(13, o)$ and $N(14, o)$ and we can compare these with the actual values. The result is that there is the agreement with the exception of the value of $N(n, 8)$, which is smaller than expected. With respect to the presence of values with exceptional number of occurences 16 and 32, the single conclusion can only be that these values have to be the result of merging of 2 or 4 values with the number of occurences 8 into a single value. After accounting for these exceptional values we get the complete agreement between the

expected and calculated values of $N(n, 8)$.

From formulae (9) we easily obtain the formula for the total number of positive values $N(n)$ for $n \geq 4$:

$$N(n) = \sum\nolimits_o N(n, o) = 68 \cdot 4^{n-2} - 20, \tag{10}$$

and we can calculate the corrected values of the quantity N in Table 2: for $n = 13$ we obtain $N = 285\,212\,652$ and for $n = 14$ we have $N = 1\,140\,850\,668$. Similarly, we can calculate the total number of occurences of all positive values $K_+(n)$ for $n \geq 4$:

$$K_+(n) = \sum\nolimits_o o\, N(n, o) = 30 \cdot 4^n - 9 \cdot 2^n + 2, \tag{11}$$

and we can compare this with the total number of coordinates of all vertices $K(n)$ which is

$$K(n) = 3\,(10 \cdot 4^n + 2). \tag{12}$$

Their difference $K(n) - K_+(n) = 9 \cdot 2^n + 4$ is the total number of occurences of zero in the coordinates of all vertices (for $n \geq 4$).

So far we have considered only the properties of the values of coordinates of vertices in the calculated data. Now we have to mention also the intrinsic properties of the vectors of vertices, namely their mutual position, which is expressed by the values of the scalar product of neighbouring vertices (for explanation see NS, Sections 2, 5, 7). As we have described in Section 2, in the case of the vertices of degree 0, the components of radius-vectors of these vertices were chosen in order to minimize the number of different values of the squared lengths of these vectors and the number of different values of the scalar product of vectors of neighbouring vertices. For degrees $n > 0$ we have only aimed to correct the components of radius-vectors of the vertices in order to make their squared length as close to 1.0 as possible. This was because the including of the testing of the scalar product of vectors of neighbouring vertices in the process of correction would be extremely difficult, both from the view of the construction of the algorithm and from the view of the computer capacity.

Therefore it was necessary, at least afterwards, to check the calculated coordinates of vertices with respect to the multiplicity of the values of the scalar product of vectors of neighbouring vertices. The first step was the calculation of these scalar products (the total number of products for degree

n is $30 \cdot 4^n$), the next step was the ordering of their values. This operation was quite difficult, not only because of the huge number of values, but also because of the distribution of these values which turned out to be extremely nonuniform; the nonuniformity increased steeply with the increasing degree.

After the ordering of the values of the scalar product it was possible to determine the groups of values that were assumed to correspond to the single theoretical value. These groups were originally defined as the sets of the successive values such that the difference between any two successive values in the set is δ_{lde4}. However, it turned out that in some cases two successive values whose difference is greater than those mentioned should belong to the same group. This is because the total number of occurences of all members of a group should be divisible by 30; this property is the consequence of the symmetries of the regular icosahedron. Therefore the definitive groups were defined as follows: if the original group satisfies the mentioned condition, it is the definitive group; if not, the definitive group consists of more successive original groups which do not satisfy this condition (it is easy to show that the choice of the definitive groups is unique).

The determination of the definitive groups has shown that for degrees $n \leq 8$ the particular groups of values were mutually well separated and the number of values within any single group was not greater than 6. This behaviour was in concordance with the presumption that each group of calculated values corresponds to a single theoretical value. However, for $n \geq 9$ there were more and more groups containing more than 6 values (the absolute maximum was 42 for degree 14), what indicates that these groups correspond to more theoretical values of scalar product.

For each group of values there was determined the number of local maxima of the distribution of the number of occurences of each value belonging to the group. It turned out that there is a clear trend of the increase of the number of the local maxima with the increasing number of values in the group. Although this cannot be an absolutely reliable criterion, it supports the view that the groups with more than 6 values very probably correspond to more theoretical values of scalar product.

The resulting statistics is presented in Table 3: for each degree n there is tabulated the number of values of the scalar product (denoted as M_i) which appear in the data with the multiplicity i (the last column contains the sum for all multiplicities $i > 10$).

The table clearly shows that in general the multiplicity of particular calculated values is small: for degrees $n \leq 11$ the number of values has a clear maximum at the multiplicity at most 4, and even for $n \geq 12$, where there are also larger multiplicities, the maximum grows only to the multiplicity 6. This demonstrates that the calculated data are sufficiently accurate.

Table 3. Number of values of scalar product of vectors of the neighbouring vertices M_i with mutiplicity i for each degree n

n	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}	$M_{>10}$
0	0	0	1	0	0	0	0	0	0	0
1	0	2	0	0	0	0	0	0	0	0
2	0	3	2	0	0	0	0	0	0	0
3	0	5	9	1	0	0	0	0	0	0
4	1	20	25	5	0	0	0	0	0	0
5	2	65	101	19	0	0	0	0	0	0
6	4	210	397	101	3	0	0	0	0	0
7	28	850	1512	391	14	0	0	0	0	0
8	81	3211	6263	1446	50	0	0	0	0	0
9	331	12793	24574	5879	244	25	7	1	0	0
10	1107	46393	93369	24227	1752	939	586	171	78	43
11	1566	59521	169018	89031	18607	35186	28825	11213	5061	4504
12	0	2887	115965	226477	100618	32194	36018	44896	23498	19272
13	0	98	47248	269645	206743	61037	46362	88297	65356	52107
14	0	0	3267	236658	383551	177530	61065	67899	102300	126384

Similarly as by the values of coordinates of vertices, an interesting result of calculation was the number of occurences of values of scalar product of the radius-vectors of neighbouring vertices. This number of occurences was calculated for each group of values (as described above) as the sum of occurences of all calculated values belonging to this group. For degrees 0 and 1 the number of occurences corresponds to its theoretical value: if we define the quantity

$$O(i) = 30 \cdot 2^i, \tag{13}$$

we find that for $n = 0$ there is one theoretical value of scalar product with $O(0)$ occurences (this corresponds to the 30 edges of the net of degree 0), while for $n = 1$ there are two theoretical values of scalar product, each with

$O(1)$ occurences. It turned out that for degrees $2 \leq n \leq 8$ the number of occurences of values belonging to each group can be equal only to one of the numbers $O(i)$ for $1 \leq i \leq n$. The number of groups with the total number of occurences equal to $O(i)$ is $P(n,i)$, where

$$\begin{aligned} i = 1: &\quad P(n,1) = 2^{n-1}, \\ 1 < i < n: &\quad P(n,i) = 2 \cdot 4^{n-i}, \\ i = n: &\quad P(n,n) = 3. \end{aligned} \tag{14}$$

The existence of this empirical dependence definitively confirms that for degrees $2 \leq n \leq 8$ each group of values of scalar product corresponds to the single theoretical value of scalar product. For degrees $n \geq 9$ the formula (14) is no longer valid: speaking generally, the values of $P(n,i)$ are for lower values of i smaller than expected and for higher values of i greater than expected. Moreover, for $n \geq 11$ there appear numbers of occurences different from $O(i)$. All this supports the view that for $n \geq 9$ there are groups of values corresponding to more than one theoretical value, and therefore, the existence of groups of values with more than 6 members does not represent an inaccuracy of the calculation, but it is the consequence of the nonuniform behaviour of the theoretical values.

5. Conclusion

The above discussion shows that the calculated values of rectangular coordinates of vertices of the maximally regular net on the unit sphere differ from their theoretical value in the prevailing number of cases by at most $2\delta_{\mathrm{lde4}}$. Therefore, after rounding to the `long double` precision, the numerical values should be accurate with the possible difference of maximally δ_{ld}; this accuracy corresponds to the 20 decimal digits with the possible difference of maximally 2 in the last decimal digit.

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