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Diagonality of certain functions with respect to spherical functions

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A b s t r a c t : It is shown that certain functions of two unit vectors are diagonal when expressed as a series of spherical functions. These functions arise by decomposing the kernel of the integral equation corresponding to the Dirichlet and Neumann boundary problems for the Laplace equation for the rotational ellipsoid into a series of powers of the numerical eccentricity of ellipsoid.

Key-words: integral kernel, ellipsoid

1. Introduction

It is well known that the Dirichlet and Neumann boundary problems for the Laplace equation in some domain can be transformed to the form of integral equations. In the case that this domain is bounded by the surface of a rotational ellipsoid, it can be easily shown (see [2]) that solutions of these equations can be obtained from the solutions of a single integral equation ([2], 2.20) whose kernel is given by ([2], 3.1). This kernel can be expressed as a series of powers of the numerical eccentricity of ellipsoid ([2], 3.7, 3.6). The purpose of this work is to prove the equalities ([2], 3.20, 3.21) from which the diagonality of the integral kernel can be derived.

We use here the notation from [2]: the point of the unit sphere with angular coordinates ξ, ψ ($0 \le \xi \le \pi$, $0 \le \psi < 2\pi$) is expressed as the unit vector

$$\boldsymbol{v} = \boldsymbol{i} \sin\xi \cos\psi + \boldsymbol{j} \sin\xi \cos\psi + \boldsymbol{k} \cos\xi, \tag{1.1}$$

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where unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are mutually orthogonal; the solid angle element is $\mathrm{d}\Xi = \sin\xi \ \mathrm{d}\xi \ \mathrm{d}\psi$ (any primed quantity is obtained by replacing ξ, ψ by ξ', ψ'). Spherical functions $\mathrm{Y}_{n,m}(\boldsymbol{v})$ (they are nonzero only for $|m| \leq n$) are given by

$$\mathrm{Y}_{n,m}(\boldsymbol{v}) = \sqrt{(2n+1)\,\frac{(n-|m|)!}{(n+|m|)!}}\,\mathrm{P}_n^{|m|}(\cos\xi)\,\mathrm{e}^{\mathrm{i}m\psi} \tag{1.2}$$

and they are orthonormal on the surface of the unit sphere ([2], 3.11).

A function of two variables $f(\boldsymbol{v}, \boldsymbol{v}')$ is diagonal with respect to the basis represented by the spherical functions if there are such coefficients $f_{n,m}$ that for any n, m such that $|m| \leq n$ it holds

$$\frac{1}{4\pi}\int \mathrm{d}\Xi'\ f(\boldsymbol{v}, \boldsymbol{v}')\ \mathrm{Y}_{n,m}(\boldsymbol{v}') = f_{n,m}\,\mathrm{Y}_{n,m}(\boldsymbol{v}) \tag{1.3}$$

(integration is performed over the surface of the unit sphere). Then this function can be expressed as a series

$$f(\boldsymbol{v}, \boldsymbol{v}') = \sum\nolimits_{n\geq 0}\sum\nolimits_{|m|\leq n} f_{n,m}\,\mathrm{Y}_{n,m}(\boldsymbol{v})\ \mathrm{Y}^*_{n,m}(\boldsymbol{v}') \tag{1.4}$$

(at least formally, as this series need not to converge); the conventions for writing of sums are given in [2], Section 3. As from (1.2) it follows that $\mathrm{Y}^*_{n,m}(\boldsymbol{v}) = \mathrm{Y}_{n,-m}(\boldsymbol{v})$, if the function $f(\boldsymbol{v}, \boldsymbol{v}')$ is real, then $f^*_{n,m} = f_{n,-m}$. Moreover, if this function is symmetric with respect to the exchange of variables $\boldsymbol{v}, \boldsymbol{v}'$, then $f_{n,m} = f_{n,-m}$ and thus coefficients $f_{n,m}$ are real.

In analogy to formula ([2], 3.17), using the equality

$$\mathrm{P}_n(\boldsymbol{v}\cdot\boldsymbol{v}') = \frac{1}{2n+1}\sum\nolimits_{|m|\leq n}\mathrm{Y}_{n,m}(\boldsymbol{v})\ \mathrm{Y}^*_{n,m}(\boldsymbol{v}') \tag{1.5}$$

holding for $n \geq 0$ ([1], 3.11.2) and the Cauchy inequality, for any function $f(\boldsymbol{v}, \boldsymbol{v}')$ which is written in the form (1.4) and any coefficients F_n such that for $|m| \leq n$ it is

$$|f_{n,m}| \leq F_n, \tag{1.6}$$

we get

$$\Big|\sum\nolimits_{|m|\leq n} f_{n,m}\,\mathrm{Y}_{n,m}(\boldsymbol{v})\ \mathrm{Y}^*_{n,m}(\boldsymbol{v}')\Big|^2 \leq$$
$$\leq \Big(\sum\nolimits_{|m|\leq n}|f_{n,m}|^2\,\mathrm{Y}_{n,m}(\boldsymbol{v})\ \mathrm{Y}^*_{n,m}(\boldsymbol{v})\Big)\Big(\sum\nolimits_{|m|\leq n}\mathrm{Y}_{n,m}(\boldsymbol{v}')\ \mathrm{Y}^*_{n,m}(\boldsymbol{v}')\Big) \leq$$

$$\leq F_n^2 \left(\sum\nolimits_{|m|\leq n} \mathrm{Y}_{n,m}(\boldsymbol{v})\, \mathrm{Y}^*_{n,m}(\boldsymbol{v})\right)\left(\sum\nolimits_{|m|\leq n} \mathrm{Y}_{n,m}(\boldsymbol{v}')\, \mathrm{Y}^*_{n,m}(\boldsymbol{v}')\right) =$$

$$= F_n^2\, ((2n+1)\, \mathrm{P}_n(1))^2 = (2n+1)^2\, F_n^2 \tag{1.7}$$

and thus (for $n \geq 0$)

$$\left|\sum\nolimits_{|m|\leq n} f_{n,m}\, \mathrm{Y}_{n,m}(\boldsymbol{v})\, \mathrm{Y}^*_{n,m}(\boldsymbol{v}')\right| \leq (2n+1)\, F_n. \tag{1.8}$$

2. Demonstration of diagonality

We want to demonstrate the diagonality of functions $N_i(\boldsymbol{v}-\boldsymbol{v}')$ (for $i \geq 0$), where

$$N_i(\boldsymbol{w}) = \frac{1}{|\boldsymbol{w}|}\left(\frac{(\boldsymbol{w}\cdot\boldsymbol{k})^2}{\boldsymbol{w}^2}\right)^i. \tag{2.1}$$

Using formula (1.1) we denote for brevity

$$\nu = \boldsymbol{v}\cdot\boldsymbol{v}' = \cos\xi\cos\xi' + \sin\xi\sin\xi'\cos(\psi-\psi'), \tag{2.2}$$

$$\zeta = (\boldsymbol{v}-\boldsymbol{v}')\cdot\boldsymbol{k} = \cos\xi - \cos\xi' \tag{2.3}$$

(it is $|\nu| \leq 1$, $|\zeta| \leq 2$), so that $(\boldsymbol{v}-\boldsymbol{v}')^2 = 2(1-\nu)$ and quantities ν and ζ satisfy the inequality

$$\zeta^2 \leq 2(1-\nu). \tag{2.4}$$

Functions $N_i(\boldsymbol{v}-\boldsymbol{v}')$ can be now expressed in the form

$$N_i(\boldsymbol{v}-\boldsymbol{v}') = \frac{1}{\sqrt{2(1-\nu)}}\left(\frac{\varepsilon^2}{2(1-\nu)}\right)^i \tag{2.5}$$

showing that they are singular for $\nu = 1$. Therefore it would be advantageous to express these functions as a limit of some (suitably chosen) non-singular functions. This is straightforward in the case $i = 0$: we can write

$$N_0(\boldsymbol{v}-\boldsymbol{v}') = \frac{1}{\sqrt{2(1-\nu)}} = \lim\nolimits_{\lambda\to 1-} \frac{1}{d(\lambda,\nu)}, \tag{2.6}$$

where

$$d(\lambda,\nu)=\sqrt{1-2\lambda\nu+\lambda^2} \tag{2.7}$$

(in the next it will always be $0\le\lambda<1$). Using the well known expansion

$$\frac{1}{d(\lambda,\nu)}=\sum\nolimits_{n\ge0}\lambda^n\,\mathrm{P}_n(\nu) \tag{2.8}$$

(converging for fixed λ absolutely and uniformly with respect to variable ν) and formula (1.5) we get

$$\frac{1}{d(\lambda,\nu)}=\sum\nolimits_{n\ge0}\frac{\lambda^n}{2n+1}\sum\nolimits_{|m|\le n}\mathrm{Y}_{n,m}(\boldsymbol{v})\;\mathrm{Y}^*_{n,m}(\boldsymbol{v}') \tag{2.9}$$

what demonstrates that function $1/d(\lambda,\boldsymbol{v}{\cdot}\boldsymbol{v}')$ is diagonal with respect to the basis represented by the functions $\mathrm{Y}_{n,m}(\boldsymbol{v})$.

Therefore, we aim to show that there are such functions $N_i(\lambda,\boldsymbol{v},\boldsymbol{v}')$ (for any $i\ge0$) which are nonsingular, diagonal, symmetric with respect to the exchange of variables $\boldsymbol{v},\boldsymbol{v}'$ and it holds

$$N_i(\boldsymbol{v}-\boldsymbol{v}')=\frac{\zeta^{2i}}{d(1,\nu)^{2i+1}}=\lim\nolimits_{\lambda\to1-}N_i(\lambda,\boldsymbol{v},\boldsymbol{v}'). \tag{2.10}$$

According to the previous section these functions can be written in the form (1.4)

$$N_i(\lambda,\boldsymbol{v},\boldsymbol{v}')=\sum\nolimits_{n\ge0}\sum\nolimits_{|m|\le n}N_{i,n}(\lambda,m^2)\;\mathrm{Y}_{n,m}(\boldsymbol{v})\;\mathrm{Y}^*_{n,m}(\boldsymbol{v}') \tag{2.11}$$

and their nonsingularity can be expressed as the requirement that this series converges (for fixed λ, $0\le\lambda<1$) absolutely and uniformly. From definition (1.2) it is evident that

$$m\,\mathrm{Y}_{n,m}(\boldsymbol{v})=-\mathrm{i}\,\partial_\psi\mathrm{Y}_{n,m}(\boldsymbol{v}) \tag{2.12}$$

and if $N_{i,n}(\lambda,x)$ are well behaved functions of x (for example, if they are polynomials of x), we can write (2.11) in the form

$$\begin{aligned}N_i(\lambda,\boldsymbol{v},\boldsymbol{v}')&=\sum\nolimits_{n\ge0}\sum\nolimits_{|m|\le n}N_{i,n}(\lambda,-\partial^2_\psi)\;\mathrm{Y}_{n,m}(\boldsymbol{v})\;\mathrm{Y}^*_{n,m}(\boldsymbol{v}')=\\&=\sum\nolimits_{n\ge0}N_{i,n}(\lambda,-\partial^2_\psi)\sum\nolimits_{|m|\le n}\mathrm{Y}_{n,m}(\boldsymbol{v})\;\mathrm{Y}^*_{n,m}(\boldsymbol{v}')=\\&=\sum\nolimits_{n\ge0}(2n+1)\,N_{i,n}(\lambda,-\partial^2_\psi)\;\mathrm{P}_n(\boldsymbol{v}{\cdot}\boldsymbol{v}'),\end{aligned} \tag{2.13}$$

where we used formula (1.5).

Thus let us investigate the result of applying operator ∂_ψ^2 and its powers on a function of $\boldsymbol{v}\cdot\boldsymbol{v}'$. If we denote

$$\chi = 1 - \cos\xi\cos\xi', \tag{2.14}$$

from formulae (2.2) and (2.3) we can find

$$\partial_\psi^2\nu = -\sin\xi\sin\xi'\cos(\psi-\psi') = 1-\chi-\nu, \tag{2.15}$$

$$(\partial_\psi\nu)^2 = (-\sin\xi\sin\xi'\sin(\psi-\psi'))^2 = 1-\zeta^2-2(1-\chi)(1-\nu)-\nu^2. \tag{2.16}$$

Then we can write for some function $f(\nu)$

$$\begin{aligned}&\partial_\psi^2 f(\nu) = (\partial_\psi\nu)^2\,\partial_\nu^2 f(\nu) + (\partial_\psi^2\nu)\,\partial_\nu f(\nu) = \\ &= (1-\zeta^2-2(1-\chi)(1-\nu)-\nu^2)\,\partial_\nu^2 f(\nu) + (1-\chi-\nu)\,\partial_\nu f(\nu)\end{aligned} \tag{2.17}$$

and we see that the expression on the r.h.s. is a polynomial (of degree one at the most) of variables χ and ζ^2. This means that by applying some polynomial (of degree k) of operator ∂_ψ^2 on function $f(\nu)$ we get a polynomial of variables χ and ζ^2 of degree k at the most.

Formulae (2.5), (2.6) and (2.8) indicate that it will be suitable to express the dependence on ν in terms of function $d(\lambda,\nu)$ (λ is now a fixed parameter). According to (2.7) we have

$$\partial_\nu d(\lambda,\nu) = -\frac{\lambda}{d(\lambda,\nu)}, \qquad \partial_\nu^2 d(\lambda,\nu) = -\frac{\lambda^2}{d(\lambda,\nu)^3} \tag{2.18}$$

and

$$1-\chi-\nu = \frac{1}{2\lambda}(d(\lambda,\nu)^2 - 2\lambda\chi - (1-\lambda)^2), \tag{2.19}$$

$$\begin{aligned}&1-\zeta^2-2(1-\chi)(1-\nu)-\nu^2 = \\ &= \frac{1}{4\lambda^2}\Big(-d(\lambda,\nu)^4 + 2(2\lambda\chi+(1-\lambda)^2)\,d(\lambda,\nu)^2 - \\ &\qquad -4\lambda(1-\lambda)^2\chi - 4\lambda^2\zeta^2 - (1-\lambda)^4\Big).\end{aligned} \tag{2.20}$$

We introduce differential operator δ_x

$$\delta_x = x\partial_x \tag{2.21}$$

(which has the favourable property that $\delta_x x^\alpha = \alpha x^\alpha$) and from (2.17) we get for some function $g(\lambda, d(\lambda,\nu))$

$$
\begin{aligned}
\partial_\psi^2 g(\lambda, d(\lambda,\nu)) &= \frac{1}{4}\Big(- d(\lambda,\nu)^4 + 2(2\lambda\chi + (1-\lambda)^2)\, d(\lambda,\nu)^2 - \\
&\qquad - 4\lambda(1-\lambda)^2\chi - 4\lambda^2\zeta^2 - (1-\lambda)^4\Big) \cdot \\
&\qquad \cdot \left(\frac{1}{d(\lambda,\nu)^2}\, \partial_d^2 g(\lambda, d(\lambda,\nu)) - \frac{1}{d(\lambda,\nu)^3}\, \partial_d g(\lambda, d(\lambda,\nu)) \right) - \\
&\qquad - \frac{1}{2}(d(\lambda,\nu)^2 - 2\lambda\chi - (1-\lambda)^2)\, \frac{1}{d(\lambda,\nu)}\, \partial_d g(\lambda, d(\lambda,\nu)) = \\
&= \frac{1}{4}\Big(-1 + 2\, \frac{2\lambda\chi + (1-\lambda)^2}{d(\lambda,\nu)^2} - \frac{4\lambda(1-\lambda)^2\chi + 4\lambda^2\zeta^2 + (1-\lambda)^4}{d(\lambda,\nu)^4} \Big) \cdot \\
&\qquad \cdot (d(\lambda,\nu)^2\, \partial_d^2 g(\lambda, d(\lambda,\nu)) - d(\lambda,\nu)\, \partial_d g(\lambda, d(\lambda,\nu))) - \\
&\qquad - \frac{1}{2}\Big(1 - \frac{2\lambda\chi + (1-\lambda)^2}{d(\lambda,\nu)^2} \Big) d(\lambda,\nu)\, \partial_d g(\lambda, d(\lambda,\nu)) = \\
&= \Big(-\frac{1}{4} + \frac{P(\lambda,\chi)}{d(\lambda,\nu)^2} - \frac{Q(\lambda,\chi,\zeta)}{d(\lambda,\nu)^4} \Big)(\delta_d - 2)\delta_d g(\lambda, d(\lambda,\nu)) - \\
&\qquad - \Big(\frac{1}{2} - \frac{P(\lambda,\chi)}{d(\lambda,\nu)^2} \Big)\delta_d g(\lambda, d(\lambda,\nu)) = \\
&= \Big(-\frac{1}{4}\,\delta_d + \frac{P(\lambda,\chi)}{d(\lambda,\nu)^2}\,(\delta_d - 1) - \frac{Q(\lambda,\chi,\zeta)}{d(\lambda,\nu)^4}\,(\delta_d - 2) \Big)\delta_d g(\lambda, d(\lambda,\nu)), \qquad (2.22)
\end{aligned}
$$

where we write for brevity ∂_d instead of $\partial_{d(\lambda,\nu)}$ and δ_d instead of $\delta_{d(\lambda,\nu)}$, and we have denoted

$$P(\lambda,\chi) = \frac{1}{2}(2\lambda\chi + (1-\lambda)^2), \qquad (2.23)$$

$$Q(\lambda,\chi,\zeta) = \frac{1}{4}(4\lambda(1-\lambda)^2\chi + 4\lambda^2\zeta^2 + (1-\lambda)^4). \qquad (2.24)$$

We see that by applying an operator $K(\partial_\psi^2)$ (where $K(x)$ is a polynomial of degree k) on function $g(\lambda, d(\lambda,\nu))$ we get a polynomial of variables $P(\lambda,\chi)$ and $Q(\lambda,\chi,\zeta)$ of degree k at the most. If the function $g(\lambda, d(\lambda,\nu))$ has the form of a linear combination of only even (or only odd) powers of $d(\lambda,\nu)$, the same is true for the function $K(\partial_\psi^2)\, g(\lambda, d(\lambda,\nu))$, because operator ∂_ψ^2 can change the power of $d(\lambda,\nu)$ only by an even number (decrease it by 0, 2 or 4). Further, if the function $g(\lambda, d(\lambda,\nu))$ is a linear combination of odd powers of $d(\lambda,\nu)$, function $K(\partial_\psi^2)\, g(\lambda, d(\lambda,\nu))$ is a polynomial of variable $Q(\lambda,\chi,\zeta)$ of degree exactly k, as in this case operator ∂_ψ^2 always increases the power of $Q(\lambda, \chi, \zeta)$ by one (this is because operator $(\delta_d - 2)\delta_d$ cannot cancel any odd power of $d(\lambda, \nu)$). Particularly, if $g(\lambda, d(\lambda, \nu))$ is $d(\lambda, \nu)^{2I-1}$ (with integer I), then any term of function $K(\partial_\psi^2)\, g(\lambda, d(\lambda, \nu))$ not containing variable $P(\lambda, \chi)$ has to have the form $Q(\lambda, \chi, \zeta)^l\, d(\lambda, \nu)^{2I-4l-1}$ (apart from a numerical coefficient depending only on I and l), where $0 \leq l \leq k$. As we have $P(1, \chi) = \chi$ and $Q(1, \chi, \zeta) = \zeta^2$, in the limit $\lambda \to 1-$ any such term acquires the form $\zeta^{2l}\, d(1, \nu)^{2I-4l-1}$ (with the same numerical coefficient).

As according to formulae (2.10) and (2.13) it has to hold (for all $i \geq 0$)

$$\frac{\zeta^{2i}}{d(1, \nu)^{2i+1}} = \lim_{\lambda \to 1-} \sum_{n \geq 0} (2n+1)\, N_{i,n}(\lambda, -\partial_\psi^2)\ \mathrm{P}_n(\nu), \tag{2.25}$$

it is reasonable to adopt that all functions $N_{i,n}(\lambda, x)$ are polynomials of variable x of degree i at the most. Thus we write (for $i \geq 0$, $n \geq 0$)

$$N_{i,n}(\lambda, x) = \sum_{0 \leq j \leq i} N_{i,j,n}(\lambda)\ x^j \tag{2.26}$$

and we have for all $i \geq 0$ (at least formally)

$$\begin{aligned}
&\sum_{n \geq 0} (2n+1)\, N_{i,n}(\lambda, -\partial_\psi^2)\ \mathrm{P}_n(\nu) = \\
&= \sum_{n \geq 0} (2n+1) \sum_{0 \leq j \leq i} N_{i,j,n}(\lambda)(-\partial_\psi^2)^j \mathrm{P}_n(\nu) = \\
&= \sum_{0 \leq j \leq i} (-\partial_\psi^2)^j \sum_{n \geq 0} (2n+1)\, N_{i,j,n}(\lambda)\ \mathrm{P}_n(\nu) = \\
&= \sum_{0 \leq j \leq i} \partial_\psi^{2j} g_{i,j}(\lambda, d(\lambda, \nu)),
\end{aligned} \tag{2.27}$$

where we denoted (for $0 \leq j \leq i$)

$$g_{i,j}(\lambda, d(\lambda, \nu)) = (-1)^j \sum_{n \geq 0} (2n+1)\, N_{i,j,n}(\lambda)\ \mathrm{P}_n(\nu). \tag{2.28}$$

From formula (2.25) we get the condition ($i \geq 0$)

$$\frac{\zeta^{2i}}{d(1, \nu)^{2i+1}} = \lim_{\lambda \to 1-} \sum_{0 \leq j \leq i} \partial_\psi^{2j} g_{i,j}(\lambda, d(\lambda, \nu)) \tag{2.29}$$

and as on the l.h.s. we have a single odd power of $d(1, \nu)$ (multiplied by an even power of ζ), according to the previous discussion we adopt that (for all $0 \leq j \leq i$)

$$g_{i,j}(\lambda, z) = g_{i,j}\, z^{2i-1} \tag{2.30}$$

(coefficients $g_{i,j}$ could be chosen to depend on λ, but this is not necessary). Thus it has to hold

$$\begin{aligned}\frac{\zeta^{2i}}{d(1,\nu)^{2i+1}} &= \lim\nolimits_{\lambda\to 1-} \sum\nolimits_{0\le j\le i} g_{i,j}\, \partial_\psi^{2j} d(\lambda,\nu)^{2i-1} = \\ &= \lim\nolimits_{\lambda\to 1-} K_i(\partial_\psi^2)\, d(\lambda,\nu)^{2i-1}, \end{aligned}\tag{2.31}$$

where

$$K_i(x) = \sum\nolimits_{0\le j\le i} g_{i,j}\, x^j. \tag{2.32}$$

According to (2.22) we introduce for brevity operator $D(u, v, z, \delta_z)$ by the equality

$$D(u, v, z, \delta_z)\, g(z) = \Big(-\frac{1}{4}\,\delta_z + \frac{u}{z^2}\,(\delta_z - 1) - \frac{v}{z^4}\,(\delta_z - 2)\Big)\delta_z g(z) \tag{2.33}$$

and thus we have

$$\partial_\psi^2 g(\lambda, d(\lambda,\nu)) = D(P(\lambda,\chi), Q(\lambda,\chi,\zeta), d(\lambda,\nu), \delta_{d(\lambda,\nu)})\, g(\lambda, d(\lambda,\nu)). \tag{2.34}$$

Then we get from (2.31) the formula

$$\begin{aligned}&\frac{\zeta^{2i}}{d(1,\nu)^{2i+1}} = \\ &= \lim\nolimits_{\lambda\to 1-} K_i(D(P(\lambda,\chi), Q(\lambda,\chi,\zeta), d(\lambda,\nu), \delta_{d(\lambda,\nu)}))\, d(\lambda,\nu)^{2i-1} = \\ &= K_i(D(\chi, \zeta^2, d(1,\nu), \delta_{d(1,\nu)}))\, d(1,\nu)^{2i-1}\end{aligned}\tag{2.35}$$

or

$$v^i = z^{2i+1}\, K_i(D(u, v, z, \delta_z))\, z^{2i-1} \tag{2.36}$$

($i \ge 0$, u is arbitrary). Now it is easy to find the leading coefficient of polynomial $K_i(x)$: from definitions (2.32) and (2.33) we see that the term containing v^i can come on the r.h.s. of (2.36) only from the i-th power of the part of operator $D(u, v, z, \delta_z)$ linear in v; thus it has to hold

$$v^i = z^{2i+1}\, g_{i,i}\Big(-\frac{v}{z^4}\,(\delta_z - 2)\delta_z\Big)^i z^{2i-1}. \tag{2.37}$$

It can be easily proved by induction that for any $i \geq 0$ it is

$$\left(z^{-4}(\delta_z - 2)\delta_z\right)^i f(z) = z^{-4i}\left(\prod_{0\leq l\leq 2i-1}(\delta_z - 2l)\right) f(z) \tag{2.38}$$

(we define $\prod_{0\leq n\leq -1}\varphi(n) = 1$ and $\prod_{0\leq n\leq 0}\varphi(n) = \varphi(0)$; other conventions are the same as for sums); then we have from (2.37)

$$\begin{aligned}
1 &= z^{2i+1}\, g_{i,i}\,(-1)^i\, z^{-4i}\left(\prod_{0\leq l\leq 2i-1}(\delta_z - 2l)\right) z^{2i-1} = \\
&= g_{i,i}\,(-1)^i\, 2^{2i} \prod_{0\leq l\leq 2i-1}(i - l - 1/2) = \\
&= g_{i,i}\,(-1)^i\, 2^{2i}\, \frac{\Gamma(1/2+i)}{\Gamma(1/2-i)} = g_{i,i}\, 2^{2i}\left(\frac{\Gamma(1/2+i)}{\Gamma(1/2)}\right)^2
\end{aligned} \tag{2.39}$$

(see [1], 1.2.3) and finally

$$g_{i,i} = \frac{1}{2^{2i}}\left(\frac{\Gamma(1/2)}{\Gamma(1/2+i)}\right)^2 = g_i. \tag{2.40}$$

Now we can write polynomial $K_i(x)$ in the form

$$K_i(x) = g_i \prod_{0\leq l\leq i-1}(x + a_{i,l}) \tag{2.41}$$

and inserting in formula (2.36) we get

$$v^i = z^{2i+1}\, g_i\left(\prod_{0\leq l\leq i-1}(D(u,v,z,\delta_z) + a_{i,l})\right) z^{2i-1}. \tag{2.42}$$

According to (2.33) it is clear that in the case $i \geq 1$ the term containing z^{4i} will appear on the r.h.s. of (2.42) unless one of the constants $a_{i,l}$ is equal to $(2i-1)^2/4$. In this way it is possible to find another such values and it turns up that it could be (for all $0 \leq l \leq i-1$)

$$a_{i,l} = (2l+1)^2/4. \tag{2.43}$$

To prove that in this case formula (2.42) holds, we introduce operator $T_i(u,v,z,\delta_z)$ by the formula ($i \geq 0$)

$$T_i(u,v,z,\delta_z)\, g(z) = \left(\prod_{0\leq l\leq i-1}(D(u,v,z,\delta_z) + (l+1/2)^2)\right) z^{2i} g(z) \tag{2.44}$$

and thus (2.42) gets the form

$$v^i = z^{2i+1}\, g_i\, T_i(u,v,z,\delta_z)\, z^{-1}. \tag{2.45}$$

We have $T_0(u,v,z,\delta_z)=1$ and for any $i\geq 0$ we get using (2.33)

$$T_{i+1}(u,v,z,\delta_z)\,g(z) = \Big(\frac{1}{4}((2i{+}1)^2-\delta_z^2)+\frac{u}{z^2}\,(\delta_z{-}1)\delta_z - \\ -\frac{v}{z^4}\,(\delta_z{-}2)\delta_z\Big)\,T_i(u,v,z,\delta_z)\,z^2 g(z). \tag{2.46}$$

Now we write operator $T_i(u,v,z,\delta_z)$ in the form

$$T_i(u,v,z,\delta_z)=\sum\nolimits_{|j|\leq i} z^{2j}\,T_{i,j}(u,v,\delta_z) \tag{2.47}$$

(it is evident that there are no other powers of z) and for $i=0$ we have $T_{0,0}(u,v,w)=1$ (we define $T_{i,j}(u,v,w)$ to be zero if $|j|>i$). Inserting (2.47) in (2.46) we get

$$\sum\nolimits_{|j|\leq i+1} z^{2j}\,T_{i+1,j}(u,v,\delta_z) = \\ =\sum\nolimits_{|j|\leq i} z^{2j+2}\Big(\frac{1}{4}((2i{+}1)^2-(\delta_z{+}2j{+}2)^2)+\frac{u}{z^2}\,(\delta_z{+}2j{+}1)(\delta_z{+}2j{+}2) - \\ -\frac{v}{z^4}\,(\delta_z{+}2j)(\delta_z{+}2j{+}2)\Big)T_{i,j}(u,v,\delta_z{+}2) = \\ =\sum\nolimits_{|j|\leq i+1} z^{2j}\,\Big(\frac{1}{4}((2i{+}1)^2-(\delta_z{+}2j)^2)\,T_{i,j-1}(u,v,\delta_z{+}2)+ \\ +u(\delta_z{+}2j{+}1)(\delta_z{+}2j{+}2)\,T_{i,j}(u,v,\delta_z{+}2)- \\ -v(\delta_z{+}2j{+}2)(\delta_z{+}2j{+}4)\,T_{i,j+1}(u,v,\delta_z{+}2)\Big) \tag{2.48}$$

and thus we have

$$T_{i+1,j}(u,v,w) = -\frac{1}{4}(w{-}2i{+}2j{-}1)(w{+}2i{+}2j{+}1)\,T_{i,j-1}(u,v,w{+}2)+ \\ +u(w{+}2j{+}1)(w{+}2j{+}2)\,T_{i,j}(u,v,w{+}2)- \\ -v(w{+}2j{+}2)(w{+}2j{+}4)\,T_{i,j+1}(u,v,w{+}2). \tag{2.49}$$

Calculation of functions $T_{i,j}(u,v,w)$ for a few low values of i shows that it could be suitable to write

$$T_{i,j}(u,v,w)=\frac{\Gamma(w/2{+}i{+}1)}{\Gamma(w/2{+}j{+}1)}\,\frac{\Gamma(w/2{+}i{+}j{+}1/2)}{\Gamma(w/2{+}1/2)}\,S_{i,j}(u,v,w) \tag{2.50}$$

(thus also $S_{i,j}(u,v,w)$ is zero for $|j| > i$). Inserting in (2.49) we get the condition

$$(w/2{+}1/2)\,S_{i+1,j}(u,v,w) = -\,(w/2{-}i{+}j{-}1/2)\,S_{i,j-1}(u,v,w{+}2)\,+ \\ +\,4u(w/2{+}j{+}1/2)\,S_{i,j}(u,v,w{+}2)\,- \\ -\,4v(w/2{+}i{+}j{+}3/2)\,S_{i,j+1}(u,v,w{+}2). \quad (2.51)$$

According to (2.50) we have $S_{0,0}(u,v,w) = 1$ and calculation of functions $S_{i,j}(u,v,w)$ for a few low values of i shows that they do not depend on w. Therefore we write

$$S_{i,j}(u,v,w) = S_{i,j}(u,v) \quad (2.52)$$

and we obtain from (2.51) two conditions for functions $S_{i,j}(u,v)$:

$$S_{i+1,j}(u,v) = -\,S_{i,j-1}(u,v) + 4u\,S_{i,j}(u,v) - 4v\,S_{i,j+1}(u,v), \quad (2.53)$$

$$0 = (i{-}j{+}1)\,S_{i,j-1}(u,v) + 4uj\,S_{i,j}(u,v) - 4v(i{+}j{+}1)\,S_{i,j+1}(u,v). \quad (2.54)$$

We multiply these equations with y^j (where y is a parameter) and sum with respect to index j. If we define the functions

$$S_i(u,v,y) = \sum\nolimits_{|j|\le i} S_{i,j}(u,v)\;y^j, \quad (2.55)$$

we get for them the conditions

$$S_{i+1}(u,v,y) = \Big(-\,y + 4u - \frac{4v}{y}\Big)S_i(u,v,y), \quad (2.56)$$

$$0 = \Big(y(i{-}\delta_y) + 4u\delta_y - \frac{4v}{y}\,(i{+}\delta_y)\Big)S_i(u,v,y) \quad (2.57)$$

and we have $S_0(u,v,y) = 1$. Then we get from (2.56)

$$S_i(u,v,y) = \Big(-\,y + 4u - \frac{4v}{y}\Big)^i \quad (2.58)$$

and thus

$$\Big(y(i{-}\delta_y) + 4u\delta_y - \frac{4v}{y}\,(i{+}\delta_y)\Big)S_i(u,v,y) = \\ = \Big[\Big(-\,y + 4u - \frac{4v}{y}\Big)\delta_y + \Big(y - \frac{4v}{y}\Big)i\Big]\Big(-\,y + 4u - \frac{4v}{y}\Big)^i = 0, \quad (2.59)$$

so that condition (2.57) is satisfied. This means that our presumption (2.52) was correct and we can get the explicit expression for functions $S_{i,j}(u,v)$ from (2.55) and (2.58) by using the binomial formula twice. For us it will be sufficient to know that for fixed i,u,v ($i \geq 0$), functions $S_{i,j}(u,v)$ are finite and that

$$S_{i,-i}(u,v) = (-4v)^i. \tag{2.60}$$

We can now write the explicit form of operator $T_i(u,v,z,\delta_z)$. Using formulae (2.47), (2.50), (2.52) and denoting for brevity

$$Q_{i,j}(w) = \frac{\Gamma(w/2+i+1)}{\Gamma(w/2+j+1)}\,\frac{\Gamma(w/2+i+j+1/2)}{\Gamma(w/2+1/2)} \tag{2.61}$$

(note that for $|j| \leq i$ it is a polynomial of variable w) we get

$$T_i(u,v,z,\delta_z) = \sum\nolimits_{|j|\leq i} S_{i,j}(u,v)\; z^{2j}\, Q_{i,j}(\delta_z) \tag{2.62}$$

and thus

$$T_i(u,v,z,\delta_z)\, z^w = \sum\nolimits_{|j|\leq i} S_{i,j}(u,v)\; Q_{i,j}(w)\; z^{w+2j}. \tag{2.63}$$

For us only the value $w = -1$ is interesting: in this case we see from (2.61) that $Q_{i,j}(-1)$ is for $|j| \leq i$ nonzero only if $j = -i$ (as $\Gamma(\alpha)$ is infinite only for $\alpha = -n$, $n \geq 0$). Then we get using (2.60) and (2.61) (see also [1], 1.2.3) for $i \geq 0$

$$\begin{aligned} T_i(u,v,z,\delta_z)\, z^{-1} &= \sum\nolimits_{|j|\leq i} S_{i,j}(u,v)\; Q_{i,j}(-1)\; z^{2j-1} = \\ &= S_{i,-i}(u,v)\; Q_{i,-i}(-1)\; z^{-2i-1} = \\ &= (-4v)^i\, \frac{\Gamma(1/2+i)}{\Gamma(1/2-i)}\; z^{-2i-1} = 2^{2i}\left(\frac{\Gamma(1/2+i)}{\Gamma(1/2)}\right)^2 v^i\; z^{-2i-1} \end{aligned} \tag{2.64}$$

and according to (2.40) this proves formula (2.45). Therefore also formula (2.42) with (2.43) is valid and the same is true for all previous forms of this condition up to formula (2.29). Particularly, from (2.36) and (2.34) we get ($i \geq 0$)

$$K_i(\partial_\psi^2)\, d(\lambda,\nu)^{2i-1} = \frac{Q(\lambda,\chi,\zeta)^i}{d(\lambda,\nu)^{2i+1}}. \tag{2.65}$$

3. Expression as a series of spherical functions

To proceed further we have to express function $d(\lambda,\nu)^{2i-1}$ as a series of Legendre polynomials; we write

$$d(\lambda,\nu)^{2i-1} = \sum\nolimits_{n\geq 0} b_{i,n}(\lambda)\ \lambda^n\ \mathrm{P}_n(\nu) \tag{3.1}$$

and according to (2.8) it is $b_{0,n}(\lambda) = 1$. In order to find coefficients $b_{i,n}(\lambda)$ for $i \geq 1$, we use the formula

$$\delta_\lambda d(\lambda,\nu) = \frac{1}{2}\left(d(\lambda,\nu) - \frac{1-\lambda^2}{d(\lambda,\nu)}\right) \tag{3.2}$$

from which we easily derive the equality

$$(i+1/2-\delta_\lambda)\, d(\lambda,\nu)^{2i+1} = (i+1/2)(1-\lambda^2)\, d(\lambda,\nu)^{2i-1} \tag{3.3}$$

and inserting (3.1) in (3.3) we get the condition ($n \geq 0$)

$$(i+1/2-n-\delta_\lambda)\, b_{i+1,n}(\lambda) = (i+1/2)(1-\lambda^2)\, b_{i,n}(\lambda). \tag{3.4}$$

From (3.1) it is easily possible to show (by induction with respect to i) that coefficients $b_{i,n}(\lambda)$ can be expressed in the form of a series of λ; then it follows from (3.4) that these coefficients are polynomials of $1-\lambda^2$ of degree i (as the operator on the l.h.s. of (3.4) cannot cancel any integer power of λ). Therefore we write ($n \geq 0$)

$$b_{i,n}(\lambda) = \sum\nolimits_{0\leq j\leq i} b_{i,j,n}\ (1-\lambda^2)^j \tag{3.5}$$

and we have $b_{0,0,n} = 1$ (we define $b_{i,j,n}$ to be zero if not $0 \leq j \leq i$). Then we get inserting (3.5) in (3.4)

$$\begin{aligned}
&(i+1/2-n-\delta_\lambda) \sum\nolimits_{0\leq j\leq i+1} b_{i+1,j,n}\ (1-\lambda^2)^j = \\
&= \sum\nolimits_{0\leq j\leq i+1} b_{i+1,j,n}\ ((i+1/2-n-2j)(1-\lambda^2)^j + 2j(1-\lambda^2)^{j-1}) = \\
&= \sum\nolimits_{0\leq j\leq i+1} ((i+1/2-n-2j)\, b_{i+1,j,n} + 2(j+1)\, b_{i+1,j+1,n})(1-\lambda^2)^j = \\
&= (i+1/2) \sum\nolimits_{0\leq j\leq i+1} b_{i,j-1,n}\ (1-\lambda^2)^j
\end{aligned} \tag{3.6}$$

and thus

$$(i+1/2-n-2j)\, b_{i+1,j,n} + 2(j+1)\, b_{i+1,j+1,n} = (i+1/2)\, b_{i,j-1,n}. \tag{3.7}$$

Calculation of coefficients $b_{i,j,n}$ for a few low values of i indicates that we can write

$$b_{i,j,n} = (-1)^i\, (n+1/2)\, \frac{\Gamma(i+1/2)}{\Gamma(1/2)}\, \frac{\Gamma(n+j-i+1/2)}{\Gamma(n+i+3/2)}\, B_{i,j,n} \tag{3.8}$$

($B_{0,0,n} = 1$ and $B_{i,j,n}$ is zero if not $0 \leq j \leq i$); inserting in (3.7) we get the formula

$$(n+2j-i-1/2)\, B_{i+1,j,n} - 2(j+1)(n+j-i-1/2)\, B_{i+1,j+1,n} =$$

$$= (n+i+3/2)\, B_{i,j-1,n}. \tag{3.9}$$

Calculation of coefficients $B_{i,j,n}$ for a few low values of i shows that they do not depend on n. Therefore we write

$$B_{i,j,n} = B_{i,j} \tag{3.10}$$

and we get from (3.9) two conditions for coefficients $B_{i,j}$

$$B_{i+1,j} - 2(j+1)\, B_{i+1,j+1} = B_{i,j-1}, \tag{3.11}$$

$$j\, B_{i+1,j} - j(j+1)\, B_{i+1,j+1} = (i+1)\, B_{i,j-1}. \tag{3.12}$$

Then we easily find that

$$(i-j)\, B_{i,j} = (2i-j)(j+1)\, B_{i,j+1} \tag{3.13}$$

and we get (for $0 \leq j \leq i$)

$$B_{i,j} = \frac{(2i-j)!}{j!\,(i-j)!}. \tag{3.14}$$

This value of $B_{i,j}$ satisfies both conditions (3.11), (3.12) and thus formula (3.10) is valid. For the coefficients $b_{i,j,n}$ we get from (3.8), (3.10) and (3.14) the expression ($0 \leq j \leq i$, $n \geq 0$)

$$b_{i,j,n} = (-1)^i\, (n+1/2)\, \frac{(2i-j)!}{j!\,(i-j)!}\, \frac{\Gamma(i+1/2)}{\Gamma(1/2)}\, \frac{\Gamma(n+j-i+1/2)}{\Gamma(n+i+3/2)} \tag{3.15}$$

and function $d(\lambda, \nu)^{2i-1}$ is (for $i \geq 0$) expressed by formulae (3.1), (3.5) and (3.15). We note that (for $0 \leq j \leq i$) coefficient $b_{i,j,n}$ is a rational function of n and it contains n only in denominator which is a polynomial of n of degree $2i-j$. This means that the series on the r.h.s. of (3.1) converges (for fixed λ, $0 \le \lambda < 1$) absolutely and uniformly.

Returning to formula (2.28) we get according to (2.30) and (3.1)

$$N_{i,j,n}(\lambda) = (-1)^j \, g_{i,j} \, \frac{1}{2n+1} \, b_{i,n}(\lambda) \, \lambda^n \tag{3.16}$$

and from (2.26) using (2.32) we have

$$N_{i,n}(\lambda, x) = \frac{1}{2n+1} \, b_{i,n}(\lambda) \, \lambda^n \, K_i(-x). \tag{3.17}$$

Inserting in (2.11) we get the required diagonal expression of functions $N_i(\lambda, \boldsymbol{v}, \boldsymbol{v}')$

$$N_i(\lambda, \boldsymbol{v}, \boldsymbol{v}') =$$
$$= \sum\nolimits_{n\ge 0} \sum\nolimits_{|m|\le n} \frac{1}{2n+1} \, b_{i,n}(\lambda) \, \lambda^n \, K_i(-m^2) \, \mathrm{Y}_{n,m}(\boldsymbol{v}) \, \mathrm{Y}^*_{n,m}(\boldsymbol{v}') \tag{3.18}$$

which can be written in the form analogical to (2.13) as

$$N_i(\lambda, \boldsymbol{v}, \boldsymbol{v}') = \sum\nolimits_{n\ge 0} b_{i,n}(\lambda) \, \lambda^n \, K_i(\partial_\psi^2) \, \mathrm{P}_n(\nu). \tag{3.19}$$

According to (2.41) and (2.43) we have

$$\begin{aligned} K_i(-m^2) &= g_i \prod\nolimits_{0\le l\le i-1} ((l+1/2)^2 - m^2) = \\ &= (-1)^i \, g_i \prod\nolimits_{-i\le l\le i-1} (m+l+1/2) = \\ &= (-1)^i \, g_i \, \frac{\Gamma(m+i+1/2)}{\Gamma(m-i+1/2)} \end{aligned} \tag{3.20}$$

and thus for $|m| \le n$ it holds

$$\begin{aligned} |K_i(-m^2)| &= g_i \prod\nolimits_{0\le l\le i-1} |(l+1/2)^2 - m^2| \le \\ &\le g_i \prod\nolimits_{0\le l\le i-1} ((l+1/2)^2 + m^2) \le \\ &\le g_i \prod\nolimits_{0\le l\le i-1} ((l+1/2)^2 + n^2). \end{aligned} \tag{3.21}$$

Recalling that (1.8) follows from (1.6), we get from (3.21) using (1.5) and (2.12) the bound

$$|K_i(\partial_\psi^2)\,\mathrm{P}_n(\nu)| = \frac{1}{2n+1}\left|\sum\nolimits_{|m|\le n} K_i(-m^2)\ \mathrm{Y}_{n,m}(\boldsymbol{v})\ \mathrm{Y}^*_{n,m}(\boldsymbol{v}')\right| \le$$
$$\le g_i \prod\nolimits_{0\le l\le i-1}((l+1/2)^2+n^2); \tag{3.22}$$

as this bound is a polynomial of n, the series on the r.h.s. of (3.19) (and thus also the series on the r.h.s. of (3.18)) converges (for fixed λ, $0\le\lambda<1$) absolutely and uniformly. Therefore it is possible to exchange the order of summation and derivation on the r.h.s. of (3.19) and using (3.1) and (2.65) to get

$$N_i(\lambda,\boldsymbol{v},\boldsymbol{v}') = K_i(\partial_\psi^2)\sum\nolimits_{n\ge 0} b_{i,n}(\lambda)\ \lambda^n\ \mathrm{P}_n(\nu) =$$
$$= K_i(\partial_\psi^2)\, d(\lambda,\nu)^{2i-1} = \frac{Q(\lambda,\chi,\zeta)^i}{d(\lambda,\nu)^{2i+1}}. \tag{3.23}$$

This means that we have fulfilled all requirements we have posed on functions $N_i(\lambda,\boldsymbol{v},\boldsymbol{v}')$; particularly, according to (2.24) formula (2.10) is valid.

We still find a bound on functions $N_i(\lambda,\boldsymbol{v},\boldsymbol{v}')$: from definitions (2.3) and (2.14) we can easily derive the inequality

$$0\le 4\chi\le\zeta^2+4; \tag{3.24}$$

then using (2.4) and (2.7) we get from (2.24) (for $0\le\lambda<1$)

$$0\le 4\,Q(\lambda,\chi,\zeta) = 4\lambda(1-\lambda)^2\,\chi + 4\lambda^2\zeta^2 + (1-\lambda)^4 \le$$
$$\le (1-\lambda)^4 + 4\lambda(1-\lambda)^2 + (\lambda(1-\lambda)^2+4\lambda^2)\,\zeta^2 =$$
$$= (1+\lambda)^2\,((1-\lambda)^2+\lambda\zeta^2) \le$$
$$\le (1+\lambda)^2\,((1-\lambda)^2+2\lambda(1-\nu)) = (1+\lambda)^2\, d(\lambda,\nu)^2 \tag{3.25}$$

and we have (for $0\le\lambda<1$)

$$0\le\frac{Q(\lambda,\chi,\zeta)}{d(\lambda,\nu)^2}\le\frac{(1+\lambda)^2}{4}<1. \tag{3.26}$$

For $0\le\lambda<1$ we can also easily prove the inequality

$$d(\lambda,\nu)^2 = 1 - 2\lambda\nu + \lambda^2 \geq \frac{1}{2}\,(1-\nu) \tag{3.27}$$

(for $-1 \leq \nu \leq 0$ the l.h.s. is not smaller than 1 while the r.h.s. is not greater than 1; for $0 \leq \nu \leq 1$ the l.h.s. has as a function of λ a minimum for $\lambda = \nu$, this minimum is equal to $1-\nu^2$ and thus not smaller than the r.h.s.). Thus we obtain according to (3.23) and (2.2) (for $0 \leq \lambda < 1$) the bound

$$0 \leq N_i(\lambda, \boldsymbol{v}, \boldsymbol{v}') \leq \frac{1}{d(\lambda,\nu)} \leq \frac{2}{\sqrt{2(1-\nu)}} = \frac{2}{|\boldsymbol{v}-\boldsymbol{v}'|}. \tag{3.28}$$

Now we can write using (2.10) for any bounded function $f(\boldsymbol{v})$

$$\frac{1}{4\pi}\int \mathrm{d}\Xi'\ N_i(\boldsymbol{v}-\boldsymbol{v}')\ f(\boldsymbol{v}') = \frac{1}{4\pi}\int \mathrm{d}\Xi'\ \lim_{\lambda\to 1-} N_i(\lambda,\boldsymbol{v},\boldsymbol{v}')\ f(\boldsymbol{v}') =$$
$$= \lim_{\lambda\to 1-} \frac{1}{4\pi}\int \mathrm{d}\Xi'\ N_i(\lambda,\boldsymbol{v},\boldsymbol{v}')\ f(\boldsymbol{v}'), \tag{3.29}$$

as according to (3.28) functions $N_i(\lambda, \boldsymbol{v}, \boldsymbol{v}')$ are uniformly (with respect to λ) bounded by an integrable function. Then we get using (3.18), (3.5), (3.15), (3.20), (2.40) and the orthonormality of spherical functions ($i \geq 0$, $|m| \leq n$)

$$\frac{1}{4\pi}\int \mathrm{d}\Xi'\ N_i(\boldsymbol{v}-\boldsymbol{v}')\ \mathrm{Y}_{n,m}(\boldsymbol{v}') = \lim_{\lambda\to 1-}\frac{1}{4\pi}\int \mathrm{d}\Xi'\ N_i(\lambda,\boldsymbol{v},\boldsymbol{v}')\ \mathrm{Y}_{n,m}(\boldsymbol{v}') =$$
$$= \lim_{\lambda\to 1-}\frac{1}{2n+1}\ b_{i,n}(\lambda)\ \lambda^n\, K_i(-m^2)\ \mathrm{Y}_{n,m}(\boldsymbol{v}) =$$
$$= \frac{1}{2n+1}\ b_{i,0,n}\, K_i(-m^2)\ \mathrm{Y}_{n,m}(\boldsymbol{v}) =$$
$$= \frac{1}{2^{2i+1}}\ \frac{(2i)!}{i!}\ \frac{\Gamma(1/2)}{\Gamma(i+1/2)}\ \frac{\Gamma(n-i+1/2)}{\Gamma(n+i+3/2)}\ \frac{\Gamma(m+i+1/2)}{\Gamma(m-i+1/2)}\ \mathrm{Y}_{n,m}(\boldsymbol{v}) =$$
$$= \frac{1}{2}\ \frac{\Gamma(n-i+1/2)}{\Gamma(n+i+3/2)}\ \frac{\Gamma(m+i+1/2)}{\Gamma(m-i+1/2)}\ \mathrm{Y}_{n,m}(\boldsymbol{v}) \tag{3.30}$$

(see [1], 1.2.15). Thus functions $N_i(\boldsymbol{v}-\boldsymbol{v}')$ (for $i \geq 0$) are diagonal with respect to the basis represented by the functions $\mathrm{Y}_{n,m}(\boldsymbol{v})$.

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