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Solution of the Dirichlet and Neumann boundary problems for the Laplace equation for the rotational ellipsoid by the integral equation method

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Abstract: The Dirichlet and Neumann boundary problems for the Laplace equation for the rotational ellipsoid are expressed in the form of integral equations. These equations are solved by the diagonalization of their integral kernels.

1. Introduction

It is well known that the Dirichlet and Neumann boundary problems for the Laplace equation in some domain can be transformed to the form of an integral equation by expressing the potential as the single-layer, resp. double-layer potential. In the case of the Dirichlet problem we write the potential in the form

$$V(\boldsymbol{r}) = \frac{1}{4\pi} \int_S \mathrm{d}\boldsymbol{\sigma} \cdot \frac{\boldsymbol{s}-\boldsymbol{r}}{|\boldsymbol{s}-\boldsymbol{r}|^3}\, u_1(\boldsymbol{s}) + \frac{1}{4\pi} \int_S \mathrm{d}\sigma\, \frac{1}{|\boldsymbol{s}-\boldsymbol{r}|}\, u_0(\boldsymbol{s}), \tag{1.1}$$

where $\boldsymbol{r}$ is the radius vector of an arbitrary point in space, S is the (sufficiently smooth) boundary of the bounded domain D, $\boldsymbol{s}$ is the radius vector of an arbitrary point of the surface S and $\mathrm{d}\boldsymbol{\sigma}$ is the vector surface element of the surface S at the point $\boldsymbol{s}$ which can be written as

$$\mathrm{d}\boldsymbol{\sigma} = \mathrm{d}\sigma\, \boldsymbol{n}(\boldsymbol{s}), \tag{1.2}$$

where $\mathrm{d}\sigma$ is the corresponding scalar surface element and $\boldsymbol{n}(\boldsymbol{s})$ is the unit vector of the external normal to the surface S at the point $\boldsymbol{s}$. The second term on the r.h.s. of (1.1) is necessary only in the case of the exterior

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problem (as the first term alone cannot describe a potential decreasing at infinity as $1/|\boldsymbol{r}|$) and the function $u_0(\boldsymbol{s})$ can be chosen arbitrarily with the single condition that its integral over the whole surface has to have a given value. If we denote the inner (resp. outer) limit of the function $f(\boldsymbol{r})$ at the point $\boldsymbol{s}$ of the surface S as $[f(\boldsymbol{s})]_{\text{int}}$ (respectively $[f(\boldsymbol{s})]_{\text{ext}}$), we get from (1.1) the well-known formula

$$[V(\boldsymbol{s})]_{\text{int}} = \frac{1}{2}\, u_1(\boldsymbol{s}) + \frac{1}{4\pi} \int_S \mathrm{d}\sigma'\, \boldsymbol{n}(\boldsymbol{s}') \cdot \frac{\boldsymbol{s}' - \boldsymbol{s}}{|\boldsymbol{s}' - \boldsymbol{s}|^3}\, u_1(\boldsymbol{s}') \tag{1.3}$$

for the interior problem and

$$\begin{aligned}[V(\boldsymbol{s})]_{\text{ext}} = &- \frac{1}{2}\, u_1(\boldsymbol{s}) + \frac{1}{4\pi} \int_S \mathrm{d}\sigma'\, \boldsymbol{n}(\boldsymbol{s}') \cdot \frac{\boldsymbol{s}' - \boldsymbol{s}}{|\boldsymbol{s}' - \boldsymbol{s}|^3}\, u_1(\boldsymbol{s}') + \\ &+ \frac{1}{4\pi} \int_S \mathrm{d}\sigma'\, \frac{1}{|\boldsymbol{s}' - \boldsymbol{s}|}\, u_0(\boldsymbol{s}')\end{aligned} \tag{1.4}$$

for the exterior problem ($\boldsymbol{s}'$ is again a point of the surface S). Similarly in the case of the Neumann problem we have for the potential the expression

$$V(\boldsymbol{r}) = \frac{1}{4\pi} \int_S \mathrm{d}\sigma\, \frac{1}{|\boldsymbol{s} - \boldsymbol{r}|}\, u_2(\boldsymbol{s}) \tag{1.5}$$

and if we denote the normal component of the inner (resp. outer) limit of a gradient of the function $f(\boldsymbol{r})$ at the point $\boldsymbol{s}$ as $[\nu_{\boldsymbol{s}} f(\boldsymbol{s})]_{\text{int}} = \boldsymbol{n}(\boldsymbol{s}) \cdot [\nabla_{\boldsymbol{s}} f(\boldsymbol{s})]_{\text{int}}$ (resp. as $[\nu_{\boldsymbol{s}} f(\boldsymbol{s})]_{\text{ext}} = \boldsymbol{n}(\boldsymbol{s}) \cdot [\nabla_{\boldsymbol{s}} f(\boldsymbol{s})]_{\text{ext}}$), we get from (1.5) the formula

$$[\nu_{\boldsymbol{s}} V(\boldsymbol{s})]_{\text{int}} = \frac{1}{2}\, u_2(\boldsymbol{s}) + \frac{1}{4\pi} \int_S \mathrm{d}\sigma'\, \boldsymbol{n}(\boldsymbol{s}) \cdot \frac{\boldsymbol{s}' - \boldsymbol{s}}{|\boldsymbol{s}' - \boldsymbol{s}|^3}\, u_2(\boldsymbol{s}') \tag{1.6}$$

for the interior problem and

$$[\nu_{\boldsymbol{s}} V(\boldsymbol{s})]_{\text{ext}} = -\frac{1}{2}\, u_2(\boldsymbol{s}) + \frac{1}{4\pi} \int_S \mathrm{d}\sigma'\, \boldsymbol{n}(\boldsymbol{s}) \cdot \frac{\boldsymbol{s}' - \boldsymbol{s}}{|\boldsymbol{s}' - \boldsymbol{s}|^3}\, u_2(\boldsymbol{s}') \tag{1.7}$$

for the exterior problem. We denote for brevity

$$n(\boldsymbol{s}, \boldsymbol{s}') = 2\, \boldsymbol{n}(\boldsymbol{s}) \cdot \frac{\boldsymbol{s} - \boldsymbol{s}'}{|\boldsymbol{s} - \boldsymbol{s}'|^3}; \tag{1.8}$$

as it can be shown that for a sufficiently smooth surface S the function $n(\boldsymbol{s}, \boldsymbol{s}')$ can be majorized by a function $c/|\boldsymbol{s} - \boldsymbol{s}'|$ (where c is a suitable constant), the integrals in the formulae (1.3), (1.4), (1.6) and (1.7) exist. It can be also easily shown that it holds

$$\frac{1}{4\pi}\int_S \mathrm{d}\sigma'\; n(\boldsymbol{s}',\boldsymbol{s}) = 1. \tag{1.9}$$

The integral formulae (1.3), (1.4) are special cases of the integral equation

$$g_1(\boldsymbol{s}) = f_1(\mu,\boldsymbol{s}) + \frac{\mu}{4\pi}\int_S \mathrm{d}\sigma'\; n(\boldsymbol{s}',\boldsymbol{s})\; f_1(\mu,\boldsymbol{s}'); \tag{1.10}$$

writing $f_1(\mu,\boldsymbol{s}) = u_1(\boldsymbol{s})$ we get for $\mu = 1$ and

$$g_1(\boldsymbol{s}) = 2\; [V(\boldsymbol{s})]_{\mathrm{int}} \tag{1.11}$$

the formula (1.3), while for $\mu = -1$ and

$$g_1(\boldsymbol{s}) = -\,2\left([V(\boldsymbol{s})]_{\mathrm{ext}} - \frac{1}{4\pi}\int_S \mathrm{d}\sigma'\; \frac{1}{|\boldsymbol{s}'-\boldsymbol{s}|}\; u_0(\boldsymbol{s}')\right) \tag{1.12}$$

the formula (1.4). Similarly the integral formulae (1.6), (1.7) are special cases of the integral equation

$$g_2(\boldsymbol{s}) = f_2(\mu,\boldsymbol{s}) + \frac{\mu}{4\pi}\int_S \mathrm{d}\sigma'\; n(\boldsymbol{s},\boldsymbol{s}')\; f_2(\mu,\boldsymbol{s}'); \tag{1.13}$$

writing $f_2(\mu,\boldsymbol{s}) = u_2(\boldsymbol{s})$ we get for $\mu = -1$ and

$$g_2(\boldsymbol{s}) = 2\; [\nu_{\boldsymbol{s}} V(\boldsymbol{s})]_{\mathrm{int}} \tag{1.14}$$

the formula (1.6), while for $\mu = 1$ and

$$g_2(\boldsymbol{s}) = -\,2\; [\nu_{\boldsymbol{s}} V(\boldsymbol{s})]_{\mathrm{ext}} \tag{1.15}$$

the formula (1.7). The integral equations (1.10), (1.13) are mutually conjugate.

2. Integral equations for the rotational ellipsoid

Our aim is to solve the integral equations (1.10), (1.13) in the case that the surface S has the form of a rotational ellipsoid. We introduce the rectangular coordinate system with the origin in the centre of the ellipsoid and the base vectors $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ so that the unit vector $\boldsymbol{k}$ is parallel to the rotational axis. In the corresponding spherical coordinate system with the coordinates r,ϑ,φ we can write the radius vector $\boldsymbol{r}$ as

$$\boldsymbol{r} = r\,(\boldsymbol{i}\sin\vartheta\cos\varphi + \boldsymbol{j}\sin\vartheta\sin\varphi + \boldsymbol{k}\cos\vartheta). \tag{2.1}$$

Let the equatorial (resp. polar) radius of the ellipsoid be a (resp. b); we shall assume that $b \le a$ and write

$$b = a\sqrt{1-\varepsilon^2}, \tag{2.2}$$

where $0 \le \varepsilon < 1$. Then the surface S is defined in the parametric form as

$$\boldsymbol{r} = \boldsymbol{s}(\xi,\psi), \tag{2.3}$$

where

$$\boldsymbol{s}(\xi,\psi) = a\,(\boldsymbol{i}\sin\xi\cos\psi + \boldsymbol{j}\sin\xi\sin\psi + \boldsymbol{k}\sqrt{1-\varepsilon^2}\cos\xi) \tag{2.4}$$

and $0 \le \xi \le \pi$, $0 \le \psi < 2\pi$ (according to our convention from the previous section the vector $\boldsymbol{r}$ satisfying (2.3) will be written as $\boldsymbol{s}$). Then we get comparing (2.1) with (2.3) and (2.4) the parametric expression of the spherical coordinates of a point of the surface S

$$\begin{aligned} r &= a\sqrt{1-\varepsilon^2(\cos\xi)^2}, \\ \vartheta &= \arccos\frac{\sqrt{1-\varepsilon^2}\cos\xi}{\sqrt{1-\varepsilon^2(\cos\xi)^2}}, \\ \varphi &= \psi. \end{aligned} \tag{2.5}$$

The vector surface element of the surface S can be calculated by the formula

$$\mathrm{d}\boldsymbol{\sigma} = \eta\,\partial_\xi\boldsymbol{s}(\xi,\psi) \times \partial_\psi\boldsymbol{s}(\xi,\psi)\,\mathrm{d}\xi\mathrm{d}\psi, \tag{2.6}$$

where η is chosen so that the element has the required orientation ($\eta^2 = 1$). Using (2.4) we get that $\eta = 1$ and

$$\mathrm{d}\boldsymbol{\sigma} = a^2\,\boldsymbol{o}(\xi,\psi)\,\mathrm{d}\Xi, \tag{2.7}$$

where

$$\boldsymbol{o}(\xi,\psi) = \boldsymbol{i}\sqrt{1-\varepsilon^2}\sin\xi\cos\psi + \boldsymbol{j}\sqrt{1-\varepsilon^2}\sin\xi\sin\psi + \boldsymbol{k}\cos\xi \tag{2.8}$$

and

$$\mathrm{d}\Xi = \sin\xi\,\mathrm{d}\xi\mathrm{d}\psi. \tag{2.9}$$

Further we introduce the unit vector

$$\boldsymbol{v} = \boldsymbol{i}\sin\xi\cos\psi + \boldsymbol{j}\sin\xi\sin\psi + \boldsymbol{k}\cos\xi \tag{2.10}$$

and we can write any function $f(\xi,\psi)$ shortly as $f(\boldsymbol{v})$. Denoting

$$k(\boldsymbol{v}) = |\boldsymbol{o}(\boldsymbol{v})| = \sqrt{1-\varepsilon^2(\sin\xi)^2} \tag{2.11}$$

we get from (2.7) according to (1.2)

$$\mathrm{d}\sigma = a^2\, k(\boldsymbol{v})\ \mathrm{d}\Xi \tag{2.12}$$

and

$$\boldsymbol{n}(\boldsymbol{s}) = \boldsymbol{n}(\boldsymbol{s}(\boldsymbol{v})) = \frac{\boldsymbol{o}(\boldsymbol{v})}{k(\boldsymbol{v})}. \tag{2.13}$$

For arbitrary vector $\boldsymbol{w}$ we define the function

$$l(\boldsymbol{w}) = \sqrt{\boldsymbol{w}^2 - \varepsilon^2(\boldsymbol{w}\cdot\boldsymbol{k})^2}; \tag{2.14}$$

then using the formulae (2.4), (2.14), (2.13) and (2.8) we can calculate

$$|\boldsymbol{s}-\boldsymbol{s}'| = |\boldsymbol{s}(\boldsymbol{v})-\boldsymbol{s}(\boldsymbol{v}')| = a\ l(\boldsymbol{v}-\boldsymbol{v}'), \tag{2.15}$$

$$\boldsymbol{n}(\boldsymbol{s})\cdot(\boldsymbol{s}-\boldsymbol{s}') = \boldsymbol{n}(\boldsymbol{s}(\boldsymbol{v}))\cdot(\boldsymbol{s}(\boldsymbol{v})-\boldsymbol{s}(\boldsymbol{v}')) = a\sqrt{1-\varepsilon^2}\frac{(1-\boldsymbol{v}\cdot\boldsymbol{v}')}{k(\boldsymbol{v})} \tag{2.16}$$

and from the formula (1.8) we have

$$\begin{aligned} n(\boldsymbol{s},\boldsymbol{s}') = n(\boldsymbol{s}(\boldsymbol{v}),\boldsymbol{s}(\boldsymbol{v}')) &= \frac{\sqrt{1-\varepsilon^2}}{a^2}\ \frac{2\,(1-\boldsymbol{v}\cdot\boldsymbol{v}')}{k(\boldsymbol{v})\ l(\boldsymbol{v}-\boldsymbol{v}')^3} = \\ &= \frac{\sqrt{1-\varepsilon^2}}{a^2}\ \frac{(\boldsymbol{v}-\boldsymbol{v}')^2}{k(\boldsymbol{v})\ l(\boldsymbol{v}-\boldsymbol{v}')^3}. \end{aligned} \tag{2.17}$$

Inserting the expressions (2.12) and (2.17) in the integral equations (1.10) and (1.13) we get

$$g_1(\boldsymbol{s}(\boldsymbol{v})) = f_1(\mu,\boldsymbol{s}(\boldsymbol{v})) + \mu\sqrt{1-\varepsilon^2}\ \frac{1}{4\pi}\int \mathrm{d}\Xi'\ \frac{(\boldsymbol{v}-\boldsymbol{v}')^2}{l(\boldsymbol{v}-\boldsymbol{v}')^3}\ f_1(\mu,\boldsymbol{s}(\boldsymbol{v}')), \tag{2.18}$$

$$\begin{aligned} k(\boldsymbol{v})\ g_2(\boldsymbol{s}(\boldsymbol{v})) = k(\boldsymbol{v})\ &f_2(\mu,\boldsymbol{s}(\boldsymbol{v})) + \\ &+\mu\sqrt{1-\varepsilon^2}\ \frac{1}{4\pi}\int \mathrm{d}\Xi'\ \frac{(\boldsymbol{v}-\boldsymbol{v}')^2}{l(\boldsymbol{v}-\boldsymbol{v}')^3}\ k(\boldsymbol{v}')\ f_2(\mu,\boldsymbol{s}(\boldsymbol{v}')) \end{aligned} \tag{2.19}$$

and we see that these equations are special cases of the single equation

$$g(\boldsymbol{v}) = f(\mu,\boldsymbol{v}) + \mu\sqrt{1-\varepsilon^2}\ \frac{1}{4\pi}\int \mathrm{d}\Xi'\ N(\boldsymbol{v}-\boldsymbol{v}')\ f(\mu,\boldsymbol{v}'), \tag{2.20}$$

where

$$N(\boldsymbol{w}) = \frac{\boldsymbol{w}^2}{l(\boldsymbol{w})^3}. \tag{2.21}$$

As it holds $(\boldsymbol{w}\cdot\boldsymbol{k})^2 \le \boldsymbol{w}^2$, we can derive from (2.14) the inequality

$$\sqrt{1-\varepsilon^2}\ |\boldsymbol{w}|\ \le\ l(\boldsymbol{w})\ \le\ |\boldsymbol{w}| \tag{2.22}$$

and from (2.21) then follows the bound

$$\frac{1}{|\boldsymbol{w}|}\ \le\ N(\boldsymbol{w})\ \le\ \frac{1}{\sqrt{1-\varepsilon^2}^{\,3}}\ \frac{1}{|\boldsymbol{w}|} \tag{2.23}$$

showing that the kernel of the integral equation (2.20) is weakly singular.

It will also be useful to express explicitly the integral term in the formula (1.12): using the formulae (2.12) and (2.15) we get

$$\frac{1}{4\pi}\int_S \mathrm{d}\sigma'\ \frac{1}{|\boldsymbol{s}'-\boldsymbol{s}|}\ u_0(\boldsymbol{s}') = a\ \frac{1}{4\pi}\int \mathrm{d}\Xi'\ \frac{1}{l(\boldsymbol{v}-\boldsymbol{v}')}\ k(\boldsymbol{v}')\ u_0(\boldsymbol{s}(\boldsymbol{v}')) \tag{2.24}$$

and putting

$$u_0(\boldsymbol{s}(\boldsymbol{v})) = \frac{1}{a\,k(\boldsymbol{v})}\ u_0, \tag{2.25}$$

where u_0 is a constant (to be defined later) we get

$$\frac{1}{4\pi}\int_S \mathrm{d}\sigma'\ \frac{1}{|\boldsymbol{s}'-\boldsymbol{s}|}\ u_0(\boldsymbol{s}') = u_0\ \frac{1}{4\pi}\int \mathrm{d}\Xi'\ \frac{1}{l(\boldsymbol{v}-\boldsymbol{v}')}. \tag{2.26}$$

3. Diagonalization of the integral kernel

We first write the function $N(\boldsymbol{w})$ in the form of a power series with respect to the parameter ε^2. According to (2.21) and (2.14) we have

$$N(\boldsymbol{w}) = \frac{1}{|\boldsymbol{w}|}\left(1-\frac{\varepsilon^2(\boldsymbol{w}\cdot\boldsymbol{k})^2}{\boldsymbol{w}^2}\right)^{-3/2} \tag{3.1}$$

and we can use the binomial formula

$$(1-x)^{-\alpha} = \sum\nolimits_{i\geq 0} \binom{-\alpha}{i} (-x)^i = \sum\nolimits_{i\geq 0} \frac{1}{i!} \frac{\Gamma(i+\alpha)}{\Gamma(\alpha)} x^i \tag{3.2}$$

(see [1], 1.2.4), where the series on the r.h.s. converges absolutely and uniformly for $|x| \leq c$ (for any $0 < c < 1$). Then we get from (3.1) the expression

$$N(\boldsymbol{w}) = \frac{1}{|\boldsymbol{w}|} \sum\nolimits_{i\geq 0} \frac{1}{i!} \frac{\Gamma(i+3/2)}{\Gamma(3/2)} \left(\frac{\varepsilon^2(\boldsymbol{w}\cdot\boldsymbol{k})^2}{\boldsymbol{w}^2}\right)^i, \tag{3.3}$$

where the series on the r.h.s. converges absolutely and uniformly (as we have $0 \leq \varepsilon < 1$). Note that we adopted the following abbreviate notation of sums: $\sum_{n\geq k}$ (resp. $\sum_{n\leq k}$) is the summation over n (the first variable in the condition) from k to ∞ (resp. from $-\infty$ to k) and $\sum_{k\leq n\leq l}$ is the summation over n (the middle variable in the condition) from k to l when $k \leq l$ and zero otherwise ($\sum_{|n|\leq k}$ is the abbreviation of $\sum_{-k\leq n\leq k}$).

Analogically to (3.1) and (3.3) we can easily derive the formulae

$$\frac{1}{l(\boldsymbol{w})} = \frac{1}{|\boldsymbol{w}|} \left(1 - \frac{\varepsilon^2(\boldsymbol{w}\cdot\boldsymbol{k})^2}{\boldsymbol{w}^2}\right)^{-1/2} \tag{3.4}$$

and

$$\frac{1}{l(\boldsymbol{w})} = \frac{1}{|\boldsymbol{w}|} \sum\nolimits_{i\geq 0} \frac{1}{i!} \frac{\Gamma(i+1/2)}{\Gamma(1/2)} \left(\frac{\varepsilon^2(\boldsymbol{w}\cdot\boldsymbol{k})^2}{\boldsymbol{w}^2}\right)^i, \tag{3.5}$$

where the series on the r.h.s converges absolutely and uniformly.

We denote for brevity

$$N_i(\boldsymbol{w}) = \frac{1}{|\boldsymbol{w}|} \left(\frac{(\boldsymbol{w}\cdot\boldsymbol{k})^2}{\boldsymbol{w}^2}\right)^i \tag{3.6}$$

and thus

$$N(\boldsymbol{w}) = \sum\nolimits_{i\geq 0} \frac{1}{i!} \frac{\Gamma(i+3/2)}{\Gamma(3/2)} \varepsilon^{2i} N_i(\boldsymbol{w}). \tag{3.7}$$

Then we have for any bounded function $f(\boldsymbol{v})$

$$\begin{aligned} &\frac{1}{4\pi} \int \mathrm{d}\Xi' \, N(\boldsymbol{v}-\boldsymbol{v}') \, f(\boldsymbol{v}') = \\ &= \sum\nolimits_{i\geq 0} \frac{1}{i!} \frac{\Gamma(i+3/2)}{\Gamma(3/2)} \varepsilon^{2i} \frac{1}{4\pi} \int \mathrm{d}\Xi' \, N_i(\boldsymbol{v}-\boldsymbol{v}') \, f(\boldsymbol{v}'), \end{aligned} \tag{3.8}$$

as according to (3.6) the series expressing the function $N(\boldsymbol{v}-\boldsymbol{v}')$ can be majorized by an integrable function (see also (2.23)).

The form of the functions $N_i(\boldsymbol{v}-\boldsymbol{v}')$ indicates that it will be useful to express the function $f(\boldsymbol{v})$ as a series of spherical functions. These are defined by the formula

$$\mathrm{Y}_{n,m}(\boldsymbol{v}) = C_{n,m}\,\mathrm{P}_n^{|m|}(\cos\xi)\,\mathrm{e}^{\mathrm{i}m\psi} \tag{3.9}$$

(see (2.10) for the connection between $\boldsymbol{v}$ and ξ,ψ), where $\mathrm{P}_n^m(u)$ are the associated Legendre functions and

$$\begin{aligned} n \geq |m| : &\qquad C_{n,m} = \sqrt{(2n+1)\,\frac{(n-|m|)!}{(n+|m|)!}}\,, \\ n < |m| : &\qquad C_{n,m} = 0 \end{aligned} \tag{3.10}$$

are the normalization constants. The basic property of the spherical functions is their orthonormality: for $n \geq |m|, n' \geq |m'|$ it holds

$$\frac{1}{4\pi}\int \mathrm{d}\Xi\ \mathrm{Y}^*_{n,m}(\boldsymbol{v})\ \mathrm{Y}_{n',m'}(\boldsymbol{v}) = \delta_{n,n'}\,\delta_{m,m'} \tag{3.11}$$

([1], 3.12.19, 3.12.21), where the asterisk denotes the complex conjugation. Spherical functions form a complete system of functions on the sphere: any function $f(\boldsymbol{v})$ continuous on the sphere can be expressed as a convergent series

$$f(\boldsymbol{v}) = \sum\nolimits_{n\geq 0}\sum\nolimits_{|m|\leq n} f_{n,m}\,\mathrm{Y}_{n,m}(\boldsymbol{v}), \tag{3.12}$$

where the coefficients $f_{n,m}$ are given by

$$f_{n,m} = \frac{1}{4\pi}\int \mathrm{d}\Xi\ f(\boldsymbol{v})\,\mathrm{Y}^*_{n,m}(\boldsymbol{v}). \tag{3.13}$$

As from the definition (3.9), (3.10) it follows that

$$\mathrm{Y}^*_{n,m}(\boldsymbol{v}) = \mathrm{Y}_{n,-m}(\boldsymbol{v}), \tag{3.14}$$

the coefficients $f_{n,m}$ of a real function $f(\boldsymbol{v})$ satisfy the equality

$$f^*_{n,m} = f_{n,-m}. \tag{3.15}$$

For us it is useful the decomposition

$$\mathrm{P}_n(\boldsymbol{v}\cdot\boldsymbol{v}') = \frac{1}{2n+1} \sum\nolimits_{|m|\le n} \mathrm{Y}_{n,m}(\boldsymbol{v})\ \mathrm{Y}^*_{n,m}(\boldsymbol{v}') \tag{3.16}$$

holding for $n \ge 0$ ([1], 3.11.2). Using the Cauchy inequality and the formula (3.16) we get for any function $f(\boldsymbol{v})$ which is written in the form (3.12)

$$\begin{aligned} &\Big|\sum\nolimits_{|m|\le n} f_{n,m}\,\mathrm{Y}_{n,m}(\boldsymbol{v})\Big|^2 \le \\ &\le \Big(\sum\nolimits_{|m|\le n} |f_{n,m}|^2\Big)\Big(\sum\nolimits_{|m|\le n} \mathrm{Y}_{n,m}(\boldsymbol{v})\ \mathrm{Y}^*_{n,m}(\boldsymbol{v})\Big) = \\ &= \Big(\sum\nolimits_{|m|\le n} |f_{n,m}|^2\Big)\,(2n+1)\,\mathrm{P}_n(1) = (2n+1)\sum\nolimits_{|m|\le n} |f_{n,m}|^2 \end{aligned} \tag{3.17}$$

and introducing the norm defined by

$$\|f_{n,*}\| = \sqrt{(2n+1)\sum\nolimits_{|m|\le n} |f_{n,m}|^2} \tag{3.18}$$

we get the bound

$$|f(\boldsymbol{v})| \le \sum\nolimits_{n\ge 0} \Big|\sum\nolimits_{|m|\le n} f_{n,m}\,\mathrm{Y}_{n,m}(\boldsymbol{v})\Big| \le \sum\nolimits_{n\ge 0} \|f_{n,*}\|. \tag{3.19}$$

For any function $f(\boldsymbol{v})$ given in the form (3.12) and such that the series on the r.h.s. of (3.19) converges (such a function will be called for brevity as belonging to the class S), the series on the r.h.s. of (3.12) converges absolutely and uniformly.

The main result of this work is the following one: for $i \ge 0$, $|m| \le n$ it holds

$$\frac{1}{4\pi}\int \mathrm{d}\Xi'\ N_i(\boldsymbol{v}-\boldsymbol{v}')\ \mathrm{Y}_{n,m}(\boldsymbol{v}') = N_{i,n,m}\,\mathrm{Y}_{n,m}(\boldsymbol{v}), \tag{3.20}$$

where

$$N_{i,n,m} = \frac{1}{2}\,\frac{\Gamma(n-i+1/2)}{\Gamma(n+i+3/2)}\,\frac{\Gamma(m+i+1/2)}{\Gamma(m-i+1/2)}. \tag{3.21}$$

The formula (3.20) shows that the integral kernels $N_i(\boldsymbol{v}-\boldsymbol{v}')$ are diagonal with respect to the basis represented by the spherical functions. Unfortunately, the derivation of the formulae (3.20) and (3.21) is too long to be presented here; therefore it will be published in the next volume of this periodical.

From the formula (3.20) we get for any function $f(\boldsymbol{v})$ that belongs to the class S

$$\frac{1}{4\pi}\int \mathrm{d}\Xi'\, N_i(\boldsymbol{v}-\boldsymbol{v}')\, f(\boldsymbol{v}') = \sum\nolimits_{n\geq 0}\sum\nolimits_{|m|\leq n} N_{i,n,m}\, f_{n,m}\, \mathrm{Y}_{n,m}(\boldsymbol{v}), \qquad (3.22)$$

as the functions $N_i(\boldsymbol{v}-\boldsymbol{v}')$ are according to (3.6) for any $i \geq 0$ bounded by the integrable function $1/|\boldsymbol{v}-\boldsymbol{v}'|$ and the series (3.12) expressing the function $f(\boldsymbol{v})$ is according to (3.19) majorized by a convergent series with constant members. Now we investigate the convergence of the series on the r.h.s. of (3.22): we denote for brevity

$$q_{i,k} = \left|\frac{\Gamma(k+i+1/2)}{\Gamma(k-i+1/2)}\right|; \qquad (3.23)$$

then $q_{i,k}$ is always non-zero and $q_{i,-k} = q_{i,k}$ (see [1], 1.2.3); as for $i \geq 0$, $k \geq 0$ it holds

$$q_{i,k+1} = \left|\frac{\Gamma(k+i+3/2)}{\Gamma(k-i+3/2)}\right| = \left|\frac{\Gamma(k+i+1/2)}{\Gamma(k-i+1/2)}\right| \left|\frac{k+i+1/2}{k-i+1/2}\right| \geq q_{i,k}, \qquad (3.24)$$

$q_{i,k}$ is for $i \geq 0$, $k \geq 0$ a nondecreasing function of k. Therefore we have for $i \geq 0, |m| \leq n$

$$q_{i,m} \leq q_{i,n} \qquad (3.25)$$

and thus

$$|N_{i,n,m}| = \frac{1}{2}\,\frac{1}{n+i+1/2}\,\left|\frac{\Gamma(n-i+1/2)}{\Gamma(n+i+1/2)}\,\frac{\Gamma(m+i+1/2)}{\Gamma(m-i+1/2)}\right| \leq$$
$$\leq \frac{1}{2n+2i+1}. \qquad (3.26)$$

According to the definition (3.18) we then have (as $i \geq 0$)

$$\|N_{i,n,*}\, f_{n,*}\| \leq \frac{1}{2n+2i+1}\,\|f_{n,*}\| \leq \frac{1}{2n+1}\,\|f_{n,*}\| \qquad (3.27)$$

and therefore, if the function $f(\boldsymbol{v})$ belongs to the class S, then the function defined by the r.h.s. of (3.22) also belongs to the class S and, moreover, the series expressing this function also converges uniformly with respect to i.

Now we can insert the expression (3.22) in the formula (3.8) and exchange the order of summations; we get

$$\frac{1}{4\pi}\int \mathrm{d}\Xi'\, N(\boldsymbol{v}-\boldsymbol{v}')\, f(\boldsymbol{v}') =$$
$$= \sum\nolimits_{i\geq 0}\frac{1}{i!}\,\frac{\Gamma(i+3/2)}{\Gamma(3/2)}\,\varepsilon^{2i}\sum\nolimits_{n\geq 0}\sum\nolimits_{|m|\leq n} N_{i,n,m}\, f_{n,m}\, \mathrm{Y}_{n,m}(\boldsymbol{v}) =$$
$$= \sum\nolimits_{n\geq 0}\sum\nolimits_{|m|\leq n} N_{n,m}\, f_{n,m}\, \mathrm{Y}_{n,m}(\boldsymbol{v}), \qquad (3.28)$$

where

$$N_{n,m} = \sum\nolimits_{i\geq 0} \frac{1}{i!} \frac{\Gamma(i+3/2)}{\Gamma(3/2)} \varepsilon^{2i} N_{i,n,m} =$$

$$= \frac{1}{2} \sum\nolimits_{i\geq 0} \frac{1}{i!} \frac{\Gamma(i+3/2)}{\Gamma(3/2)} \frac{\Gamma(n-i+1/2)}{\Gamma(n+i+3/2)} \frac{\Gamma(m+i+1/2)}{\Gamma(m-i+1/2)} \varepsilon^{2i} =$$

$$= (-1)^{n+m} \sum\nolimits_{i\geq 0} \frac{1}{i!} \frac{\Gamma(i+3/2)}{\Gamma(1/2)} \frac{\Gamma(i+m+1/2)}{\Gamma(i+n+3/2)} \frac{\Gamma(i-m+1/2)}{\Gamma(i-n+1/2)} \varepsilon^{2i} =$$

$$= (-1)^{n+m} \frac{1}{2} \frac{\Gamma(1/2+m)}{\Gamma(3/2+n)} \frac{\Gamma(1/2-m)}{\Gamma(1/2-n)} \cdot$$

$$\cdot\, {}_3\mathrm{F}_2(3/2,\ 1/2{+}m,\ 1/2{-}m;\ 3/2{+}n,\ 1/2{-}n;\ \varepsilon^2) =$$

$$= \frac{1}{2n+1}\, {}_3\mathrm{F}_2(3/2,\ 1/2{+}m,\ 1/2{-}m;\ 3/2{+}n,\ 1/2{-}n;\ \varepsilon^2) \tag{3.29}$$

(see [1], 1.2.3 and 4.1.1, 4.1.2).

We still express the integral on the r.h.s. of (2.26): comparing the formulae (3.3) and (3.5) we can easily show that analogically to (3.28) and (3.29) it holds

$$\frac{1}{4\pi} \int \mathrm{d}\Xi' \frac{1}{l(\boldsymbol{v}-\boldsymbol{v}')} f(\boldsymbol{v}') = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} D_{n,m}\, f_{n,m}\, \mathrm{Y}_{n,m}(\boldsymbol{v}), \tag{3.30}$$

where

$$D_{n,m} = \frac{1}{2} \sum\nolimits_{i\geq 0} \frac{1}{i!} \frac{\Gamma(i+1/2)}{\Gamma(1/2)} \frac{\Gamma(n-i+1/2)}{\Gamma(n+i+3/2)} \frac{\Gamma(m+i+1/2)}{\Gamma(m-i+1/2)} \varepsilon^{2i} =$$

$$= \frac{1}{2n+1}\, {}_3\mathrm{F}_2(1/2,\ 1/2{+}m,\ 1/2{-}m;\ 3/2{+}n,\ 1/2{-}n;\ \varepsilon^2). \tag{3.31}$$

According to (3.9)–(3.13) we get for $f(\boldsymbol{v}) = 1 = \mathrm{Y}_{0,0}(\boldsymbol{v})$ the formula

$$\frac{1}{4\pi} \int \mathrm{d}\Xi' \frac{1}{l(\boldsymbol{v}-\boldsymbol{v}')} = D_{0,0} = {}_3\mathrm{F}_2(1/2,\ 1/2,\ 1/2;\ 3/2,\ 1/2;\ \varepsilon^2) =$$

$$= {}_2\mathrm{F}_1(1/2,\ 1/2;\ 3/2;\ \varepsilon^2) = \frac{\arcsin\varepsilon}{\varepsilon} \tag{3.32}$$

(see [1], 2.8.1, 2.8.13).

4. Solution of the integral equation

We return to the integral equation (2.20) and write the functions $g(\boldsymbol{v})$ and $f(\mu, \boldsymbol{v})$ in the form (3.12)

$$g(\boldsymbol{v}) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} g_{n,m}\, \mathrm{Y}_{n,m}(\boldsymbol{v}), \tag{4.1}$$

$$f(\mu, \boldsymbol{v}) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} f_{n,m}(\mu)\, \mathrm{Y}_{n,m}(\boldsymbol{v}) \tag{4.2}$$

and we propose that these functions belong to the class S. Then we get from (2.20) using (4.1), (4.2) and (3.28)

$$\sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} g_{n,m}\, \mathrm{Y}_{n,m}(\boldsymbol{v}) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} f_{n,m}(\mu)\, \mathrm{Y}_{n,m}(\boldsymbol{v}) + \\ + \mu\sqrt{1-\varepsilon^2}\, \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} N_{n,m}\, f_{n,m}(\mu)\, \mathrm{Y}_{n,m}(\boldsymbol{v}) \tag{4.3}$$

and denoting

$$\Lambda_{n,m} = \sqrt{1-\varepsilon^2}\, N_{n,m} \tag{4.4}$$

we finally have (for $|m| \leq n$)

$$g_{n,m} = (1 + \mu\, \Lambda_{n,m})\, f_{n,m}(\mu). \tag{4.5}$$

From the definitions (4.4), (3.29) and the inequality (3.26) we easily get the bound ($|m| \leq n$)

$$|\Lambda_{n,m}| \leq \sqrt{1-\varepsilon^2}\, \frac{1}{2} \sum\nolimits_{i\geq 0} \frac{1}{i!} \frac{\Gamma(i+3/2)}{\Gamma(3/2)} \frac{\varepsilon^{2i}}{n+i+1/2} = \Lambda_{n,n} \tag{4.6}$$

and we see that $\Lambda_{n,n}$ is (strongly) decreasing function of n. As for $n \geq 0$, $i \geq 0$ it holds

$$\frac{1}{(2n+1)(i+1/2)} \leq \frac{1}{n+i+1/2} \leq \frac{1}{i+1/2}, \tag{4.7}$$

using (3.2) we get for $n \geq 0$

$$\frac{1}{2n+1} \frac{1}{\sqrt{1-\varepsilon^2}} \leq \frac{1}{2} \sum\nolimits_{i\geq 0} \frac{1}{i!} \frac{\Gamma(i+3/2)}{\Gamma(3/2)} \frac{\varepsilon^{2i}}{n+i+1/2} \leq \frac{1}{\sqrt{1-\varepsilon^2}} \tag{4.8}$$

and thus for $n \geq 0$ we have

$$\frac{1}{2n+1} \leq \Lambda_{n,n} \leq 1. \tag{4.9}$$

Therefore it holds $\Lambda_{0,0} = 1$ and all other coefficients $\Lambda_{n,m}$ (for $|m| \leq n$) are absolutely smaller than 1.

Now it is easy to find the solution of Eq. (4.5): in the case $-1 < \mu \leq 1$ we have simply

$$f_{n,m}(\mu) = \frac{1}{1 + \mu\,\Lambda_{n,m}}\; g_{n,m} \tag{4.10}$$

for all $|m| \leq n$, whereas in the case $\mu = -1$ it has to be $g_{0,0} = 0$, the value of $f_{0,0}(-1)$ is arbitrary and the formula (4.10) holds for $|m| \leq n$, $n \geq 1$. In both cases it is evident that if the function $g(\boldsymbol{v})$ belongs to the class S, the same is holding for the function $f(\mu, \boldsymbol{v})$.

5. Solution of the boundary problems

We can now apply the results of the previous section to the four boundary problems described in Section 1. In the case of the Dirichlet problem we compare Eqs (2.20), (2.18) and (1.10) and we get

$$f(\mu, \boldsymbol{v}) = f_1(\mu, \boldsymbol{s}(\boldsymbol{v})) = u_1(\boldsymbol{s}(\boldsymbol{v})), \tag{5.1}$$

where the function $\boldsymbol{s}(\boldsymbol{v})$ is given by the formula (2.4). For the interior problem we have $\mu = 1$ and using (1.11) we get

$$g(\boldsymbol{v}) = g_1(\boldsymbol{s}(\boldsymbol{v})) = 2\,[V(\boldsymbol{s}(\boldsymbol{v}))]_{\mathrm{int}}. \tag{5.2}$$

Then we have according to (4.1), (3.12) and (3.13) (for $|m| \leq n$)

$$g_{n,m} = \frac{1}{4\pi} \int \mathrm{d}\Xi\; 2\,[V(\boldsymbol{s}(\boldsymbol{v}))]_{\mathrm{int}}\; \mathrm{Y}^{*}_{n,m}(\boldsymbol{v}) \tag{5.3}$$

and using (5.1), (4.2) and (4.10) we get

$$u_1(\boldsymbol{s}(\boldsymbol{v})) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} \frac{1}{1 + \Lambda_{n,m}}\; g_{n,m}\, \mathrm{Y}_{n,m}(\boldsymbol{v}), \tag{5.4}$$

where $\Lambda_{n,m}$ is given by (4.4) and (3.29)

$$\Lambda_{n,m} =$$
$$= \frac{1}{2n+1}\; \sqrt{1 - \varepsilon^2}\; {}_3\mathrm{F}_2(3/2,\; 1/2{+}m,\; 1/2{-}m;\; 3/2{+}n,\; 1/2{-}n;\; \varepsilon^2). \tag{5.5}$$

The resulting (interior) potential can be written according to the formulae (1.1) and (2.7) in the form

$$V(\boldsymbol{r}) = \frac{1}{4\pi} \int \mathrm{d}\Xi\; a^2\, \boldsymbol{o}(\boldsymbol{v}) \cdot \frac{\boldsymbol{s}(\boldsymbol{v}) - \boldsymbol{r}}{|\boldsymbol{s}(\boldsymbol{v}) - \boldsymbol{r}|^3}\; u_1(\boldsymbol{s}(\boldsymbol{v})). \tag{5.6}$$

For the exterior problem we have $\mu = -1$ and using (1.12), (2.26) and (3.32) we get

$$g(\boldsymbol{v}) = g_1(\boldsymbol{s}(\boldsymbol{v})) = -\,2\,\left([V(\boldsymbol{s}(\boldsymbol{v}))]_{\text{ext}} - u_0\,\frac{\arcsin\varepsilon}{\varepsilon}\right) \tag{5.7}$$

and thus

$$g_{0,0} = \frac{1}{4\pi}\int \mathrm{d}\Xi\,(-\,2)\,\left([V(\boldsymbol{s}(\boldsymbol{v}))]_{\text{ext}} - u_0\,\frac{\arcsin\varepsilon}{\varepsilon}\right). \tag{5.8}$$

As it has to hold $g_{0,0} = 0$, we get for the constant u_0

$$u_0 = \frac{\varepsilon}{\arcsin\varepsilon}\,\frac{1}{4\pi}\int \mathrm{d}\Xi\,[V(\boldsymbol{s}(\boldsymbol{v}))]_{\text{ext}} \tag{5.9}$$

and then we have (for $|m| \leq n$, $n \geq 1$)

$$g_{n,m} = \frac{1}{4\pi}\int \mathrm{d}\Xi\,(-\,2)\,[V(\boldsymbol{s}(\boldsymbol{v}))]_{\text{ext}}\,\mathrm{Y}^*_{n,m}(\boldsymbol{v}) \tag{5.10}$$

and finally

$$u_1(\boldsymbol{s}(\boldsymbol{v})) = \sum\nolimits_{n\geq 1}\sum\nolimits_{|m|\leq n}\frac{1}{1-\Lambda_{n,m}}\,g_{n,m}\,\mathrm{Y}_{n,m}(\boldsymbol{v}). \tag{5.11}$$

We could add to the expression on the r.h.s. a constant term (corresponding to the member of the sum with $n = 0$), but such a term has no effect on the resulting (exterior) potential that is given according to (1.1), (2.7), (2.12) and (2.25) by the formula

$$\begin{aligned} V(\boldsymbol{r}) = {} & \frac{1}{4\pi}\int \mathrm{d}\Xi\,a^2\,\boldsymbol{o}(\boldsymbol{v})\cdot\frac{\boldsymbol{s}(\boldsymbol{v})-\boldsymbol{r}}{|\boldsymbol{s}(\boldsymbol{v})-\boldsymbol{r}|^3}\,u_1(\boldsymbol{s}(\boldsymbol{v})) + \\ & + \frac{1}{4\pi}\int \mathrm{d}\Xi\,a\,\frac{1}{|\boldsymbol{s}(\boldsymbol{v})-\boldsymbol{r}|}\,u_0. \end{aligned} \tag{5.12}$$

In the case of the Neumann problem we compare Eqs (2.20), (2.19) and (1.13) and we get

$$f(\mu,\boldsymbol{v}) = k(\boldsymbol{v})\,f_2(\mu,\boldsymbol{s}(\boldsymbol{v})) = k(\boldsymbol{v})\,u_2(\boldsymbol{s}(\boldsymbol{v})). \tag{5.13}$$

For the interior problem we have $\mu = -1$ and using (1.14) we get

$$g(\boldsymbol{v}) = k(\boldsymbol{v})\,g_2(\boldsymbol{s}(\boldsymbol{v})) = 2\,k(\boldsymbol{v})\,[\nu_{\boldsymbol{s}(\boldsymbol{v})}V(\boldsymbol{s}(\boldsymbol{v}))]_{\text{int}}\,. \tag{5.14}$$

Then we have (for $|m| \leq n$)

$$g_{n,m} = \frac{1}{4\pi} \int \mathrm{d}\Xi \; 2\, k(\boldsymbol{v}) \; [\nu_{\boldsymbol{s}(\boldsymbol{v})} V(\boldsymbol{s}(\boldsymbol{v}))]_{\rm int} \; \mathrm{Y}^{*}_{n,m}(\boldsymbol{v}) \tag{5.15}$$

and, as it has to be $g_{0,0} = 0$, the input has to satisfy the condition

$$\frac{1}{4\pi} \int \mathrm{d}\Xi \; k(\boldsymbol{v}) \; [\nu_{\boldsymbol{s}(\boldsymbol{v})} V(\boldsymbol{s}(\boldsymbol{v}))]_{\rm int} = 0. \tag{5.16}$$

Using (5.13), (4.2) and (4.10) we get

$$u_2(\boldsymbol{s}(\boldsymbol{v})) = \frac{1}{k(\boldsymbol{v})} \Big(u_0 + \sum\nolimits_{n \geq 1} \sum\nolimits_{|m| \leq n} \frac{1}{1 - \Lambda_{n,m}} \; g_{n,m} \, \mathrm{Y}_{n,m}(\boldsymbol{v}) \Big), \tag{5.17}$$

where u_0 is arbitrary constant. For the exterior problem we have $\mu = 1$ and using (1.15) we get

$$g(\boldsymbol{v}) = k(\boldsymbol{v}) \; g_2(\boldsymbol{s}(\boldsymbol{v})) = -\, 2\, k(\boldsymbol{v}) \; [\nu_{\boldsymbol{s}(\boldsymbol{v})} V(\boldsymbol{s}(\boldsymbol{v}))]_{\rm ext} \, . \tag{5.18}$$

Then we have (for $|m| \leq n$)

$$g_{n,m} = \frac{1}{4\pi} \int \mathrm{d}\Xi \; (-\,2)\, k(\boldsymbol{v}) \; [\nu_{\boldsymbol{s}(\boldsymbol{v})} V(\boldsymbol{s}(\boldsymbol{v}))]_{\rm ext} \; \mathrm{Y}^{*}_{n,m}(\boldsymbol{v}) \tag{5.19}$$

and finally

$$u_2(\boldsymbol{s}(\boldsymbol{v})) = \frac{1}{k(\boldsymbol{v})} \sum\nolimits_{n \geq 0} \sum\nolimits_{|m| \leq n} \frac{1}{1 + \Lambda_{n,m}} \; g_{n,m} \, \mathrm{Y}_{n,m}(\boldsymbol{v}). \tag{5.20}$$

The resulting potential can be written according to the formulae (1.5) and (2.12) in the form

$$V(\boldsymbol{r}) = \frac{1}{4\pi} \int \mathrm{d}\Xi \; a^2 \, \frac{1}{|\boldsymbol{s}(\boldsymbol{v}) - \boldsymbol{r}|} \; k(\boldsymbol{v}) \; u_2(\boldsymbol{s}(\boldsymbol{v})), \tag{5.21}$$

where the function $u_2(\boldsymbol{s}(\boldsymbol{v}))$ is for the interior problem given by (5.17) and for the exterior problem by (5.20). At the end we note that for any of the mentioned boundary problems the function $\mathrm{g}(\boldsymbol{v})$ has to belong to the class S.

6. Discussion

Finally we have to compare our method of the solution of the boundary problems for the potential in the interior or exterior of a rotational ellipsoid with the standard one. It is well known that the Laplace equation can be solved in the ellipsoidal coordinates by separation of variables (see for example [1], Chapters 15 and 16) and the potential can be expressed as a series of the ellipsoidal harmonics (see for example [2], Chapter 1). For better comparison we write the formulae expressing the dependence of the spherical coordinates on the ellipsoidal coordinates (coordinates of oblate spheroid) υ, ξ, ψ in a form little different from Eq. 1–103 in [2]

$$\begin{aligned} r &= a\sqrt{(1-\varepsilon^2)\upsilon^2 + \varepsilon^2(\sin\xi)^2}, \\ \vartheta &= \arccos \frac{\sqrt{1-\varepsilon^2}\,\upsilon\cos\xi}{\sqrt{(1-\varepsilon^2)\upsilon^2 + \varepsilon^2(\sin\xi)^2}}, \\ \varphi &= \psi, \end{aligned} \tag{6.1}$$

where $\upsilon \geq 0$, $0 \leq \xi \leq \pi$, $0 \leq \psi < 2\pi$, $0 \leq \varepsilon < 1$. We see that for $\upsilon = 1$ we get from (6.1) the formula (2.5) expressing parametrically the boundary surface S; thus in the interior domain it is $0 \leq \upsilon < 1$ and in the exterior domain $\upsilon > 1$. The function $h(\boldsymbol{r})$ harmonic in the interior domain can be written in the form

$$h(\boldsymbol{r}) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} h_{n,m} \Big(\mathrm{P}_n^{|m|}(\mathrm{i}\,\kappa(\varepsilon)\,\upsilon)/\mathrm{P}_n^{|m|}(\mathrm{i}\,\kappa(\varepsilon))\Big)\mathrm{Y}_{n,m}(\boldsymbol{v}), \tag{6.2}$$

while for the function harmonic in the exterior domain it is

$$h(\boldsymbol{r}) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} h_{n,m} \Big(\mathrm{Q}_n^{|m|}(\mathrm{i}\,\kappa(\varepsilon)\,\upsilon)/\mathrm{Q}_n^{|m|}(\mathrm{i}\,\kappa(\varepsilon))\Big)\mathrm{Y}_{n,m}(\boldsymbol{v}), \tag{6.3}$$

where $\mathrm{Q}_n^{|m|}(u)$ are the Legendre functions of the second kind (see [1], Chapter 3), variable $\boldsymbol{v}$ is given by (2.10) and

$$\kappa(\varepsilon) = \frac{\sqrt{1-\varepsilon^2}}{\varepsilon}. \tag{6.4}$$

On the surface S both expressions (6.2) and (6.3) give

$$h(\boldsymbol{s}) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} h_{n,m}\,\mathrm{Y}_{n,m}(\boldsymbol{v}) \tag{6.5}$$

and thus the solution of the Dirichlet problem can be easily found (only a little more complicated it is for the Neumann problem).

From the theoretical standpoint our approach is just another form of solution of these boundary problems; however, from the practical standpoint (where the word practical means application in another branch of theory, for example in theoretical gravimetry) the situation is slightly different. In both approaches the solution is expressed by an infinite series which cannot be written in a closed form (by the author's knowledge). In our approach this is complicated by the presence of surface integrals in resulting formulae. However, it seems that our approach has two advantages: the first one is that the potential is always expressed in a form containing no singularities in the domain of its harmonicity, while in the ellipsoidal coordinates we have a singularity on a disk defined by $\vartheta = \pi/2$, $0 \leq r \leq a\,\varepsilon$ (compared to the point singularity $r = 0$ in the spherical coordinates). This singularity has no effect for the exterior problems which are mostly dealing with in the practice, but it can be inconvenient for the interior problems. The second advantage is (in the author's opinion) that there are problems which require the potential to be expressed as a surface integral. For example, in cases we have to adopt for some reason the approximate solution, such form of potential assures that it is always a harmonic function (although with only approximate boundary value). The ellipsoidal form of surface is only an approximation of the surface of real planetary bodies and therefore the solution of boundary problems for the rotational ellipsoid is in fact a first approximation for the solution of these problems for general form of surface (not very different from the ellipsoid). Therefore, our approach might be helpful for the work in this direction.

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