The maximally regular net on the sphere

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Abstract: The maximally regular division of the spherical surface into a set of domains is constructed starting from the triangular domains which are central projections of the sides of the regular icosahedron; any triangular domain is then divided into four smaller triangular domains by joining the centres of edges of the original domain by segments of great circles. The vertices of these domains represent the maximally regular net of points on the sphere. The geometrical properties of the triangular domains are investigated and it is shown that for any triangular domain, the mutual ratios of the lengths of its edges are bounded within a narrow interval. A unique and simple coding of domains and their vertices is introduced: the code of each domain and each vertex is a sequence of digits. The formulae for transformation of the code of any vertex to its Cartesian coordinates and vice versa are introduced; the neighbourhood of a point (as a set of triangular domains) is defined and the formulae allowing to find the code of any neighbouring domain from the code of the given domain (and similarly for the neighbouring vertices of the given vertex) are presented. The described construction of the net can be easily adapted to the surface of the rotational ellipsoid.

Key words: regular icosahedron, spherical triangle, domain, edge, vertex

1. Introduction

In various scientific disciplines it is often necessary to integrate numerically some quantity over the surface of a sphere or rotational ellipsoid. Numerical integration is usually performed by summing the contributions of elementary domains which are parts of the whole integration domain: the ideal case is when these elementary domains are equal. However, on the contrary to the integration over a plane, this task is more difficult in the case of the surface of a sphere: it is well known that the spherical surface cannot be divided into arbitrarily small domains of the same shape and size (see e.g. *Womersley, 2005*). Therefore, in order to find a sufficiently regular

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division of the spherical surface into arbitrarily small elementary domains, we try to start from the domains which are central projections of the sides of a regular polyhedron (which has mutually equal sides) onto the sphere and to proceed by subsequent division of these domains into the smaller ones. The vertices of all these domains will constitute the desired net of points at the spherical surface.

We shall consider this net of points at the spherical surface as maximally regular if the following two conditions are satisfied:

1. We start from a regular polyhedron which has the maximal number of sides;

2. We divide each domain into smaller domains in the simplest possible way by segments of great circles.

There exist only five regular polyhedra (see *Mathworld*); from these, the regular icosahedron has the maximal number of sides: 20. These sides are planar triangles and thus the central projections of these sides onto the spherical surface are spherical triangles. It is evident that the simplest possible way to divide any triangular domain on the spherical surface into smaller triangular domains is to construct four triangular domains by joining the centres of edges of the original domain by segments of great circles. This principle of division of domains will be therefore used for domains of arbitrarily small size.

The described way of constructing a net of points on the spherical surface is very natural and it does not represent anything new. However, on the contrary to the previous attempts known to the author (e.g. *Mayer-Guerr et al., 2004*), the main purpose of the present work is to investigate the geometrical properties of domains of arbitrarily small size and to define a suitable denotation of all domains and their vertices allowing to use the net (after adaptation to the surface of a rotational ellipsoid) as a standard reference tool for the geoscience data.

2. Vertices

We start from the regular icosahedron whose vertices lie on the unit sphere. We choose a rectangular coordinate system with origin in the centre of the icosahedron and such that one of the vertices will lie on the (positive) third axis. The vertices of the icosahedron will be denoted as follows: north pole vertex $V(00)$, south pole vertex $V(01)$, northern ring vertices $V(a0)$, southern ring vertices $V(a1)$, where a is a digit from among $\{1, 2, 3, 4, 5\}$ (according to the context, the symbol a can represent an integer denoted by a). For any such value of a we define the next value a^+ and the previous value a^- as follows:

$$1^+ = 2, \quad 2^+ = 3, \quad 3^+ = 4, \quad 4^+ = 5, \quad 5^+ = 1,$$

$$1^- = 5, \quad 2^- = 1, \quad 3^- = 2, \quad 4^- = 3, \quad 5^- = 4.$$

For any point P of the unit sphere we define the vector $\boldsymbol{v}(P)$ as the radius-vector of the point P; the radius-vectors of the vertices of the icosahedron will be shortly denoted as follows:

$$\boldsymbol{e}(00) = \boldsymbol{v}(V(00)), \quad \boldsymbol{e}(01) = \boldsymbol{v}(V(01)), \tag{1}$$

$$\boldsymbol{e}(a0) = \boldsymbol{v}(V(a0)), \quad \boldsymbol{e}(a1) = \boldsymbol{v}(V(a1)). \tag{2}$$

If we define

$$\psi_5 = \frac{\pi}{5} \tag{3}$$

and

$$c = \cos \xi_5, \quad s = \sin \xi_5, \tag{4}$$

(where $0 < \xi_5 < \pi/2$; the actual value of ξ_5 will be obtained later), we can write the vertex vectors as

$$\boldsymbol{e}(00) = (0,\, 0,\, 1), \quad \boldsymbol{e}(01) = (0,\, 0,\, -1), \tag{5}$$

$$\boldsymbol{e}(a0) = (s\, \cos{(2a-2)}\psi_5,\, s\, \sin{(2a-2)}\psi_5,\, c), \tag{6}$$

$$\boldsymbol{e}(a1) = (s\, \cos{(2a-1)}\psi_5,\, s\, \sin{(2a-1)}\psi_5,\, -c), \tag{7}$$

(note that for any integer k, $\cos{(2k+5)}\psi_5 = -\cos 2k\psi_5$, $\sin{(2k+5)}\psi_5 = -\sin 2k\psi_5$).

Each vertex of the icosahedron is a vertex of five triangular sides of the icosahedron; for any given vertex and any side whose one vertex is the given

vertex, the other two vertices of this side are the neighbouring vertices of the given vertex. Thus the neighbouring vertices are:
– for the north pole vertex $V(00)$ all vertices $V(a0)$;
– for the south pole vertex $V(01)$ all vertices $V(a1)$;
– for the northern ring vertex $V(a0)$ the vertices $V(00)$, $V(a^-0)$, $V(a^+0)$, $V(a^-1)$, $V(a1)$;
– for the southern ring vertex $V(a1)$ the vertices $V(01)$, $V(a^-1)$, $V(a^+1)$, $V(a0)$, $V(a^+0)$.
The definition of the regular icosahedron implies that the scalar product of the radius-vectors of any two neighbouring vertices of the icosahedron has to be the same. This scalar product is equal to c in the case of the neighbouring vertices of the north and south pole vertex; for the neighbouring vertices of any northern or southern ring vertex this results in the conditions

$$s^2 \cos 2\psi_5 + c^2 = c, \quad s^2 \cos \psi_5 - c^2 = c, \tag{8}$$

and therefore (as $c \neq 1$, $c \neq -1$)

$$c = \frac{\cos 2\psi_5}{1 - \cos 2\psi_5}, \quad c = \frac{\cos \psi_5}{1 + \cos \psi_5}. \tag{9}$$

In order to calculate the value of $\cos \psi_5$, we use the equality $\sin 5\psi_5 = 0$ (following from (3)) and the formula

$$\sin 5\alpha = \sin \alpha \, (16 \cos^4\alpha - 12 \cos^2\alpha + 1), \tag{10}$$

which can be easily derived using the well known formulae for the trigonometric functions of the sum of two arguments. The equation $\sin 5\alpha = 0$ is satisfied by $\alpha = k\psi_5$ for any integer k; if $1 \leq k \leq 4$, $\sin k\psi_5 \neq 0$ and the values of $\cos k\psi_5$ satisfy the equation

$$16\,u^4 - 12\,u^2 + 1 = 0. \tag{11}$$

As $\cos k\psi_5$ is for $1 \leq k \leq 4$ a decreasing function of k, $\cos \psi_5$ and $\cos 2\psi_5$ are the two largest roots of this equation, and therefore

$$\cos \psi_5 = \frac{\sqrt{5} + 1}{4}, \quad \cos 2\psi_5 = \frac{\sqrt{5} - 1}{4}. \tag{12}$$

From both equations of (9) and (4) we then get

$$c = \frac{1}{\sqrt{5}}, \quad s = \frac{2}{\sqrt{5}}, \tag{13}$$

and the formulae (6) and (7) now read

$$\boldsymbol{e}(a0) = \frac{1}{\sqrt{5}}(2\,\cos{(2a-2)\psi_5},\, 2\,\sin{(2a-2)\psi_5},\, 1), \tag{14}$$

$$\boldsymbol{e}(a1) = \frac{1}{\sqrt{5}}(2\,\cos{(2a-1)\psi_5},\, 2\,\sin{(2a-1)\psi_5},\, -1). \tag{15}$$

3. Domains

Consider the unit sphere with the same centre as the icosahedron; the sides of the icosahedron are planar triangles whose central projections on this sphere are spherical triangles. We denote these spherical triangles as $S(apq)$, where a is a digit from among $\{1,2,3,4,5\}$, while p and q are digits from among $\{0,1\}$. The vertices of the domain $S(apq)$ will be denoted as $V_i(apq)$, where the integer i ($i \in \{1,2,3\}$) is the index of the vertex. The vertices of each domain are ordered in the positive (counterclockwise) sense, when viewed from outside of the sphere; the first vertex of each domain is defined as follows. If the domain contains a polar vertex (such domain will be called a polar domain), this vertex is the first one (for this domain). Any other domain (which will be called an equatorial domain) has one vertex from either the northern or the southern ring and two vertices from either the southern or the northern ring; the former vertex is the first one (for this domain).

We introduce the following correspondence between the denotation of vertices of triangular domains and the denotation of vertices of icosahedron:

$$V_1(a00) = V(00), \quad V_2(a00) = V(a0), \quad V_3(a00) = V(a^+0), \tag{16}$$

$$V_1(a01) = V(a1), \quad V_2(a01) = V(a^+0), \quad V_3(a01) = V(a0), \tag{17}$$

$$V_1(a10) = V(01), \quad V_2(a10) = V(a^+1), \quad V_3(a10) = V(a1), \tag{18}$$

$$V_1(a11) = V(a^+0), \quad V_2(a11) = V(a1), \quad V_3(a11) = V(a^+1). \tag{19}$$

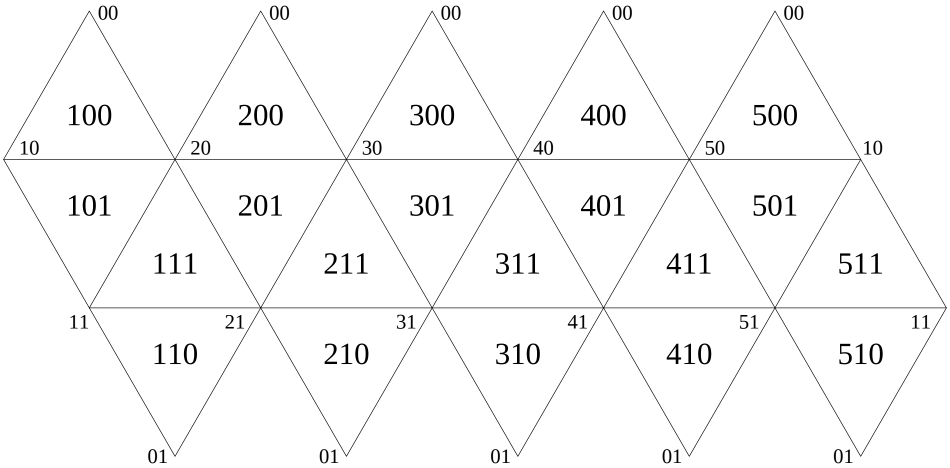

Fig. 1. The unfolded surface of the regular icosahedron. Each triangular domain $S(apq)$ is the central projection of some triangular side of the icosahedron on the unit sphere. The code of each domain apq is written in the centre of the triangular side, the code of each vertex of the icosahedron ($0p$ or ap) is written by the vertex.

The position of these domains and the vertices of the icosahedron are shown in Fig. 1.

The adopted choice of denotation can be explained as follows: the domains $S(a00)$ are the north polar ones, the domains $S(a10)$ are the south polar ones, while the domains $S(a01)$ and $S(a11)$ are the equatorial ones. Thus the polar domains have 0 as the third digit in their denotation, while the equatorial domains have 1.

We shall say that the domain has orientation o (where o is a digit from among $\{0, 1\}$) just if its first vertex is $V(0o)$ or $V(co)$ (for some $c \in \{1, 2, 3, 4, 5\}$). Thus the domains $S(a00)$ and $S(a11)$ have orientation 0, while the domains $S(a01)$ and $S(a10)$ have orientation 1; the former ones have even number of zeros in their denotation, while the latter ones have odd number of zeros (this is also the reason for the choice of values of a: they are always nonzero). We define the opposite orientation o^* to the orientation o as follows:

$0^* = 1, \quad 1^* = 0;$

we shall say that the orientations 0 and 1 are mutually opposite.

For any given domain, each domain which has a common edge with the given one is the neighbouring domain of the given one. Thus, the neighbouring domains are:

– for the north polar domain $S(a00)$ the domains $S(a01)$, $S(a^{-}00)$, $S(a^{+}00)$;
– for the south polar domain $S(a10)$ the domains $S(a11)$, $S(a^{-}10)$, $S(a^{+}10)$;
– for the equatorial domain $S(a01)$ the domains $S(a00)$, $S(a^{-}11)$, $S(a11)$;
– for the equatorial domain $S(a11)$ the domains $S(a10)$, $S(a01)$, $S(a^{+}01)$.

4. Division of domains

Consider any triangular domain at the unit sphere: this domain can be divided into four triangular domains in the most uniform way as follows. For each edge of this domain (every edge is a segment of a great circle) we find its centre and join these centres by segments of great circles. In this way we can construct iteratively new domains starting from the domains $S(apq)$.

We first present some useful definitions. For any two points of the unit sphere P_1 and P_2 which are not antipodal we define the point $C(P_1, P_2)$ as the centre of the shorter segment of the great circle joining these two points. Then we have

$$\boldsymbol{v}(C(P_1, P_2)) = \boldsymbol{c}(\boldsymbol{v}(P_1), \boldsymbol{v}(P_2)), \tag{20}$$

where the vector operation $\boldsymbol{c}(\boldsymbol{u}, \boldsymbol{v})$ is defined for any two vectors $\boldsymbol{u}$, $\boldsymbol{v}$, whose sum is not a zero vector, as follows:

$$\boldsymbol{c}(\boldsymbol{u}, \boldsymbol{v}) = \frac{\boldsymbol{u} + \boldsymbol{v}}{|\boldsymbol{u} + \boldsymbol{v}|}. \tag{21}$$

For any two points of the unit sphere P_1 and P_2 which are neither antipodal nor identical we can define the unit vector

$$\boldsymbol{w}(P_1, P_2) = \frac{\boldsymbol{v}(P_1) \times \boldsymbol{v}(P_2)}{|\boldsymbol{v}(P_1) \times \boldsymbol{v}(P_2)|}; \tag{22}$$

this vector is perpendicular to the segment joining the points P_1, P_2 and tangential to the unit sphere at each point of this segment. For any unit vector $\boldsymbol{e}$ we define the domain $H(\boldsymbol{e})$ as the hemisphere (of the unit sphere) whose pole has the radius-vector $\boldsymbol{e}$. The boundary of every hemisphere is a great circle; in the case of the domain $H(\boldsymbol{w}(P_1, P_2))$ the boundary is the great circle defined by the points P_1, P_2.

For any point P of the unit sphere, the quantity $\boldsymbol{v}(P)\cdot\boldsymbol{w}(P_1, P_2)$ describes the position of the point P relative to the great circle defined by the points P_1, P_2: its absolute value is equal to the distance of the point P from the plane of this great circle and its sign is positive just if this point belongs to the hemisphere $H(\boldsymbol{w}(P_1, P_2))$.

Now we extend our denotation of domains at the unit sphere in the following way. Any triangular domain will be denoted as $S(\Sigma)$, where Σ is a sequence of digits composed of two sequences Σ_0 and Σ_e: Σ_0 is the sequence apq as defined above, while Σ_e is a (possibly empty) sequence of digits from among $\{0, 1, 2, 3\}$; the sequence Σ_e will be empty just if the domain is one of the original domains. The vertices of the domain $S(\Sigma)$ will be denoted as $V_i(\Sigma)$, where $i \in \{1, 2, 3\}$; the assigning of indices to these vertices will be defined below.

For each vertex $V_i(\Sigma)$ ($i \in \{1, 2, 3\}$) we assign the index i to the edge of the triangular domain $S(\Sigma)$ which is opposite to this vertex. We shall say that the triple of mutually different indices (i,j,k) is ordered just if it is one of the triples (1,2,3), (2,3,1), (3,1,2). For each $i \in \{1, 2, 3\}$ we denote the centre of the i-th edge as $C_i(\Sigma)$; thus for any ordered triple (i,j,k)

$$C_i(\Sigma) = C(V_j(\Sigma), V_k(\Sigma)). \tag{23}$$

The new domains constructed by the division of the domain $S(\Sigma)$ will be denoted by $S(\Sigma r)$, where r is a digit from among $\{0, 1, 2, 3\}$. The new domains will have the following vertices (in the correct order):

$$V_1(\Sigma 0) = C_1(\Sigma), \quad V_2(\Sigma 0) = C_2(\Sigma), \quad V_3(\Sigma 0) = C_3(\Sigma), \tag{24}$$

$$V_1(\Sigma 1) = V_1(\Sigma), \quad V_2(\Sigma 1) = C_3(\Sigma), \quad V_3(\Sigma 1) = C_2(\Sigma), \tag{25}$$

$$V_1(\Sigma 2) = C_3(\Sigma), \quad V_2(\Sigma 2) = V_2(\Sigma), \quad V_3(\Sigma 2) = C_1(\Sigma), \tag{26}$$

$$V_1(\Sigma 3) = C_2(\Sigma), \quad V_2(\Sigma 3) = C_1(\Sigma), \quad V_3(\Sigma 3) = V_3(\Sigma). \tag{27}$$

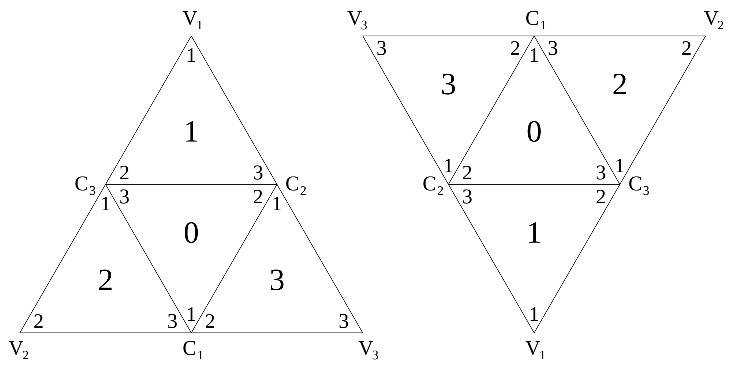

Fig. 2. The denotation of vertices and centres of edges of domains with orientation 0 (on the left) and orientation 1 (on the right). If the domain has the code Σ, the domains constructed by its division have the code Σr; the digit r is written in the centre of each smaller domain. The index i of each vertex of the smaller domains is written by the particular vertex.

We see that the vertices of new domains are ordered in the positive sense provided that the same is true for the vertices of the original domain $S(\Sigma)$ (see Fig. 2). Moreover, it is possible to extend iteratively the definition of orientation to all domains: if the original domain $S(\Sigma)$ has orientation o, the domains $S(\Sigma 1)$, $S(\Sigma 2)$, $S(\Sigma 3)$ have the same orientation o, while the domain $S(\Sigma 0)$ has the opposite orientation o^*. The presented way of denotation has also the following advantage: it is evident that the domain $S(\Sigma)$ has orientation 0 (orientation 1) just if the number of zeros in its denotation is even (odd). Thus we can define recursively the function $O(\Sigma)$ whose value is the orientation of the domain $S(\Sigma)$:

$$O(a00) = 0, \quad O(a01) = 1, \quad O(a10) = 1, \quad O(a11) = 0, \tag{28}$$

$$O(\Sigma 0) = O(\Sigma)^*, \;\; O(\Sigma 1) = O(\Sigma), \;\; O(\Sigma 2) = O(\Sigma), \;\; O(\Sigma 3) = O(\Sigma). \tag{29}$$

We still define for any domain $S(\Sigma)$ its degree $N(\Sigma)$ recursively as follows:

$$N(\Sigma_0) = 0, \quad N(\Sigma r) = N(\Sigma) + 1, \tag{30}$$

where (as above) Σ_0 is apq and $r \in \{0, 1, 2, 3\}$. The same degree $N(\Sigma)$ will be assigned to each edge and each vertex of the domain $S(\Sigma)$; thus each point which is a vertex of some domain will have infinitely many degrees (but there is always the single minimal degree for every such point). The net of all vertices with the degree N ($N \geq 0$) will be called the net of degree N (its domains will be all domains of the degree N); thus the net of degree 0 consists of all 12 vertices of the regular icosahedron and its domains are the original domains $S(apq)$.

5. Coding of vertices

The above defined denotation of triangular domains assigns an unique code for each domain. It would be therefore desirable to have an unique coding of each vertex of the net (or at least for each vertex of the net of some given degree). Each vertex has already the denotation in the form $V_i(\Sigma)$ (thus a denotation relative to the domain $S(\Sigma)$), but such denotation is not unique. Therefore we shall introduce an unique (absolute) denotation of vertices; we shall use the fact that each vertex different from any vertex of the icosahedron is the centre of some edge of some triangular domain.

We first introduce the function $e(o, r)$ ($o \in \{0, 1\}$, $r \in \{0, 1, 2, 3\}$) defined by the formulae

$$e(0,0) = 0, \quad e(0,1) = 1, \quad e(0,2) = 2, \quad e(0,3) = 3, \tag{31}$$

$$e(1,0) = 0, \quad e(1,1) = 1, \quad e(1,2) = 3, \quad e(1,3) = 2; \tag{32}$$

we see that $e(o, e(o, r)) = r$. For any domain $S(\Sigma)$ and any $i \in \{1, 2, 3\}$, the number $e(O(\Sigma), i)$ will be called the e-index of the vertex $V_i(\Sigma)$ (in the domain $S(\Sigma)$) and the number $e(O(\Sigma)^*, i)$ will be called the e-index of the i-th edge of the domain $S(\Sigma)$ and of its centre $C_i(\Sigma)$ (in the domain $S(\Sigma)$). We shall call the vertex (the edge, the centre of the edge) of the domain $S(\Sigma)$ whose e-index is 1, 2, 3, the first, left, right vertex (edge, centre of the edge) of the domain $S(\Sigma)$, respectively (see Fig. 2). The reason for introducing the e-indices is that they allow to simplify many of the subsequent considerations and formulae.

Every edge of the net is a common edge of two neighbouring domains; we shall call this edge ordinary (extraordinary) just if the orientations of these two domains are mutually opposite (equal). Of course, in fact this is not the property of the edge itself, but of the adopted way of denotation of domains. We shall now prove the following property of the indices and e-indices of the edges:

(P) For any common edge of two neighbouring domains, if this edge is ordinary, its indices in both domains are equal; if this edge is extraordinary and it has index i in one domain, then it has index $e(1, i)$ in the other domain and $i \neq 1$. For any common edge of two neighbouring domains, if this edge has e-index e in one domain, then it has e-index $e(1, e)$ in the other domain.

According to the definitions from the Section 3 (see also Fig. 1), each edge of the equatorial domains $S(a01)$ and $S(a11)$ is ordinary and its indices and e-indices have the property (P). For the edges of the polar domains $S(a00)$ and $S(a10)$, each edge with index 1 (in one of these domains) is ordinary, while each edge with indices 2 or 3 is extraordinary; for every edge, its indices and e-indices have the property (P).

Using the definitions from the Section 4 (see also Fig. 2), we easily obtain for the edges of subdomains $S(\Sigma r)$ of the domain $S(\Sigma)$ the following. Each edge of the domain $S(\Sigma 0)$ is evidently ordinary and its indices and e-indices have the property (P). For the edges of the domains $S(\Sigma r)$ $(r \neq 0)$, if the index i of the edge is equal to r, then this edge is also an edge of the domain $S(\Sigma 0)$; otherwise this edge is a part of the edge of the domain $S(\Sigma)$ with index i. As for $r \neq 0$ we have $O(\Sigma r) = O(\Sigma)$, the e-index of the i-th edge of the domain $S(\Sigma r)$ $(r \neq 0,\ i \neq r)$ is equal to the e-index of the i-th edge of the domain $S(\Sigma)$ and the former edge is ordinary (extraordinary) just if the latter edge is. Further, if the indices and e-indices of the latter edge have the property (P), the same is true for the indices and e-indices of the former edge.

We can conclude that every edge of the net satisfies (P); moreover, we see that any edge of the net can be extraordinary only if it is an edge connecting the north (south) pole vertex with some vertex of the northern (southern) ring or it is a part of such an edge.

Now we define for each domain $S(\Sigma)$ its base vertex and its base edges as follows: the base vertex will be the vertex with e-index 2 and the base edges will be the edges with e-indices 1 and 2. In other words (see Fig. 2), the base vertex is the left vertex of the domain and the base edges are the first and the left edge of the domain; the base edges are the edges whose

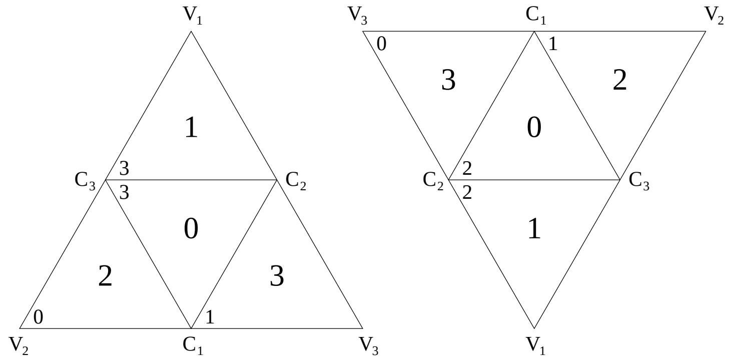

Fig. 3. The same as Fig. 2, but the direction s of the base vertex of the particular smaller domain (with respect to the base vertex of the large domain) is written by this vertex (instead of its index).

one end point is identical with the base vertex.

For each base edge of the domain $S(\Sigma)$ we assign the number called the direction (with respect to the base vertex of the domain $S(\Sigma)$): it will be equal to the index of this edge in the domain $S(\Sigma)$. Thus the direction of the base edge with e-index 1 will be 1, while the direction of the base edge with e-index 2 will be 3 (if $O(\Sigma) = 0$) and 2 (if $O(\Sigma) = 1$). Further, for each point of a base edge (note that the end points of the edge do not belong to the edge) we assign the direction equal to the direction of the edge; for the base vertex of the domain $S(\Sigma)$ we assign the direction 0 (see Fig. 3). For any base edge of the domain $S(\Sigma)$ (and any point of this edge), we shall say that the base vertex of the domain $S(\Sigma)$) is the base vertex of this base edge (and of this point of this edge). The reason for these definitions will be clear immediately.

Consider now any edge of the net and the e-indices of this edge in two neighbouring domains whose common edge is this edge; according to (P), either both e-indices of this edge are different from 1 (and thus mutually different) and then this edge is a base edge in only one of these domains (in which it has e-index 2), or both e-indices of this edge are equal to 1 and then this edge is a base edge in both domains. In the latter case the base vertices of this edge in these two domains are identical and the direction of this edge in both domains is equal to 1. Thus every edge of the net has uniquely assigned its base vertex and its direction.

Conversely, consider any vertex of the net, which is not a polar one, and let N be the minimal degree of this vertex. If $N = 0$, there are 10 such vertices and each of them is evidently the base vertex of exactly two domains of the degree 0. If $N > 0$, any such vertex is the centre of a single edge of degree $N-1$ and this edge is the common edge of two domains of degree $N-1$. As we see from Fig. 3, any vertex of the domain $S(\Sigma)$ is the base vertex of some of its subdomains $S(\Sigma r)$ just if it is the base vertex of domain $S(\Sigma)$. Any centre of an edge of the domain $S(\Sigma)$ is the base vertex of some of the subdomains $S(\Sigma r)$ just if it is a centre of a base edge of domain $S(\Sigma)$. More exactly, the centre of the edge of the domain $S(\Sigma)$ with e-index 1, 2, 3 is the base vertex of one, two, none subdomain(s) of the domain $S(\Sigma)$, respectively. Therefore, using the property (P) we obtain that any vertex which is the centre of the edge of the degree $N-1$, is the base vertex of exactly two domains of the degree N. We can conclude that any vertex of the net which is not a polar one and whose degree is N, is the base vertex of exactly two domains of the degree N' where $N' \geq N$.

These considerations imply that every edge of the net can be uniquely coded by the code of its base vertex and its direction; we shall use this code for the centre of this edge (which is always a vertex of the net).

Any vertex of the net will be denoted as $V(\mathrm{T})$, where T is a sequence of digits composed of two sequences T_0 and T_e: if the vertex is not a polar one, T_0 is the sequence ap as defined above, while T_e is a (possibly empty) sequence of digits from among $\{0, 1, 2, 3\}$; if the vertex is a polar one, T_0 is the sequence $0p$, while T_e is a (possibly empty) sequence of digits 0. If the sequence T_e is empty, the point $V(\mathrm{T})$ is identical with some vertex of the icosahedron (see Section 2). In analogy with the formula (30) we define for any vertex $V(\mathrm{T})$ its degree $N_\mathrm{v}(\mathrm{T})$ as follows:

$$N_\mathrm{v}(\mathrm{T}_0) = 0, \quad N_\mathrm{v}(\mathrm{T}s) = N_\mathrm{v}(\mathrm{T}) + 1, \tag{33}$$

where T_0 is ap or $0p$ and $s \in \{0, 1, 2, 3\}$ (do not confuse it with the real number s appearing only in the Section 2).

According to the definition of the base vertex, the e-index of a vertex and the formulae (16) – (19), each vertex $V(ap)$ is the base vertex of the domains $S(ap0)$ and $S(ap1)$. Let $V(\mathrm{T})$ be the base vertex of two neighbouring domains; then these domains have mutually opposite orientations and we can denote them as $S(\Sigma_{(0)})$, $S(\Sigma_{(1)})$, where $O(\Sigma_{(0)}) = 0$, $O(\Sigma_{(1)}) = 1$, and

$$\Sigma_{(0)} = \sigma(0, \mathrm{T}), \quad \Sigma_{(1)} = \sigma(1, \mathrm{T}), \tag{34}$$

where $\sigma(o, \mathrm{T})$ is a function which will be defined below. Then we evidently have

$$V_2(\Sigma_{(0)}) = V_3(\Sigma_{(1)}) = V(\mathrm{T}), \quad V_3(\Sigma_{(0)}) = V_2(\Sigma_{(1)}), \tag{35}$$

and using the formula (23) we define

$$V(\mathrm{T}0) = V(\mathrm{T}), \quad V(\mathrm{T}1) = C_1(\Sigma_{(0)}) = C_1(\Sigma_{(1)}), \tag{36}$$

$$V(\mathrm{T}2) = C_2(\Sigma_{(1)}), \quad V(\mathrm{T}3) = C_3(\Sigma_{(0)}). \tag{37}$$

According to the definition of the base vertex, the e-index of a vertex and the formulae (23) and (24) – (27), each of the vertices $V(\mathrm{T}s)$ (where $s \in \{0, 1, 2, 3\}$) is the base vertex of two neighbouring domains constructed by the division of the domains $S(\Sigma_{(0)})$, $S(\Sigma_{(1)})$:
– $V(\mathrm{T}0)$ is the base vertex of domains $S(\Sigma_{(0)}2)$, $S(\Sigma_{(1)}3)$;
– $V(\mathrm{T}1)$ is the base vertex of domains $S(\Sigma_{(0)}3)$, $S(\Sigma_{(1)}2)$;
– $V(\mathrm{T}2)$ is the base vertex of domains $S(\Sigma_{(1)}0)$, $S(\Sigma_{(1)}1)$;
– $V(\mathrm{T}3)$ is the base vertex of domains $S(\Sigma_{(0)}1)$, $S(\Sigma_{(0)}0)$;
(see Fig. 3). Thus the function $\sigma(o, \mathrm{T})$ can be defined recursively for any vertex $V(\mathrm{T})$ (which is not a polar one) as follows:

$$\sigma(0, a0) = a00, \quad \sigma(1, a0) = a01, \tag{38}$$

$$\sigma(0, a1) = a11, \quad \sigma(1, a1) = a10, \tag{39}$$

$$\sigma(0, \mathrm{T}0) = \sigma(0, \mathrm{T})2, \quad \sigma(1, \mathrm{T}0) = \sigma(1, \mathrm{T})3, \tag{40}$$

$$\sigma(0, \mathrm{T}1) = \sigma(0, \mathrm{T})3, \quad \sigma(1, \mathrm{T}1) = \sigma(1, \mathrm{T})2, \tag{41}$$

$$\sigma(0, \mathrm{T}2) = \sigma(1, \mathrm{T})0, \quad \sigma(1, \mathrm{T}2) = \sigma(1, \mathrm{T})1, \tag{42}$$

$$\sigma(0, \mathrm{T}3) = \sigma(0, \mathrm{T})1, \quad \sigma(1, \mathrm{T}3) = \sigma(0, \mathrm{T})0; \tag{43}$$

we see that for $o \in \{0, 1\}$ it holds

$$O(\sigma(o, \mathrm{T})) = o. \tag{44}$$

For the polar vertices $V(00)$, $V(01)$ we define using the formulae (16), (18) and (25) the following correspondence between the denotation of vertices and domains: if $V(\mathrm{T})$ is the polar vertex of the domain $S(\Sigma)$, then $V(\mathrm{T}0)$ is the polar vertex of the domain $S(\Sigma 1)$. For any vertex $V(\mathrm{T})$ which is not a polar one, its degree is according to (30), (33), (34) and (38) – (43) equal to the degree of domains $S(\Sigma_{(0)})$, $S(\Sigma_{(1)})$. Therefore the adopted denotation has the property that the vertices of domains of some degree have the same degree.

Formulae (34) and (38) – (43) allow us to determine the codes of domains $S(\Sigma_{(0)})$, $S(\Sigma_{(1)})$ from the given code of their base vertex $V(\mathrm{T})$. Now we describe the inverse procedure: determination of the code of vertices of the domain $S(\Sigma)$ (thus also of its base vertex) from the given code of this domain. We shall write the code of the vertex of the domain $S(\Sigma)$ with e-index e as $\tau_e(\Sigma)$ $(1 \le e \le 3)$; thus (see the formulae (31) – (32))

$$V_i(\Sigma) = V(\tau_{e(O(\Sigma),i)}(\Sigma)). \tag{45}$$

In order to derive the expression of the functions $\tau_e(\Sigma)$, we introduce first the functions $X_e(\Sigma)$: we shall have $X_e(\Sigma) = 0$ $(X_e(\Sigma) = 1)$ just if the edge of the domain $S(\Sigma)$ with e-index e is ordinary (extraordinary). We further introduce the following denotation of the neighbouring domains of the domain $S(\Sigma)$: the domain whose common edge with the domain $S(\Sigma)$ has (in the domain $S(\Sigma)$) the e-index 1, 2, 3, will be denoted as $S(\Sigma^*)$, $S(\Sigma^-)$, $S(\Sigma^+)$, respectively.

We evidently have $X_1(\Sigma) = 0$ and

$$X_2(ap0) = 1, \quad X_3(ap0) = 1, \quad X_2(ap1) = 0, \quad X_3(ap1) = 0, \tag{46}$$

as only polar domains have extraordinary edges. Using the formulae (23) and (24) – (27) we obtain

$$X_2(\Sigma r) = z(O(\Sigma)^*, r)X_2(\Sigma), \quad X_3(\Sigma r) = z(O(\Sigma), r)X_3(\Sigma), \tag{47}$$

where the function $z(o, r)$ $(o \in \{0, 1\}, r \in \{0, 1, 2, 3\})$ is defined by

$$z(0, 0) = 0, \quad z(0, 1) = 1, \quad z(0, 2) = 0, \quad z(0, 3) = 1, \tag{48}$$

$$z(1,0) = 0, \quad z(1,1) = 1, \quad z(1,2) = 1, \quad z(1,3) = 0. \tag{49}$$

In order to distinguish the domains and vertices whose codes have a special form, we introduce the following denotation: the sequence Σ_e (T_e) will be denoted as Σ_I (T_I), where I is a subset of the set $\{0,1,2,3\}$, if this sequence is empty or every its member belongs to the set I. Then we can easily obtain from the formulae (46) – (49) that the value of the function $X_2(\Sigma)$ will be nonzero just for the domains $S(a00\Sigma_{\{1,2\}})$ and $S(a10\Sigma_{\{1,3\}})$, while the value of the function $X_3(\Sigma)$ will be nonzero just for the domains $S(a00\Sigma_{\{1,3\}})$ and $S(a10\Sigma_{\{1,2\}})$.

According to the formulae (16) – (19), (45) and (31) – (32) we have

$$\tau_1(a00) = 00, \quad \tau_1(a01) = a1, \quad \tau_1(a10) = 01, \quad \tau_1(a11) = a^+0, \tag{50}$$

$$\tau_2(apq) = ap, \quad \tau_3(apq) = a^+p, \tag{51}$$

and using the formulae (23), (24) – (27), (45) and (31) – (32) we find the expressions for $\tau_e(\Sigma r)$ ($r \in \{0,1,2,3\}$). This is straightforward for $e = 2$: we have

$$\tau_2(\Sigma r) = \tau_2(\Sigma)s(O(\Sigma), r), \tag{52}$$

where the function $s(o,r)$ ($o \in \{0,1\}$, $r \in \{0,1,2,3\}$) is defined by

$$s(0,0) = 3, \quad s(0,1) = 3, \quad s(0,2) = 0, \quad s(0,3) = 1, \tag{53}$$

$$s(1,0) = 2, \quad s(1,1) = 2, \quad s(1,2) = 1, \quad s(1,3) = 0. \tag{54}$$

In the cases $e = 1$ and $e = 3$ we have

$$\tau_1(\Sigma 1) = \tau_1(\Sigma)0, \quad \tau_1(\Sigma e(O(\Sigma), 2)) = \tau_2(\Sigma)s(O(\Sigma), 1), \tag{55}$$

$$\tau_1(\Sigma 0) = \tau_3(\Sigma e(O(\Sigma), 2)) = \tau_2(\Sigma)1, \quad \tau_3(\Sigma e(O(\Sigma), 3)) = \tau_3(\Sigma)0. \tag{56}$$

In order to express the remaining values, we have first to find the code of the centre of the right edge of the domain $S(\Sigma)$. This centre is identical with the centre of the left edge of the domain $S(\Sigma^+)$; the base vertex of this domain is identical with the first (right) vertex of the domain $S(\Sigma)$ just if $X_3(\Sigma) = 0$ ($X_3(\Sigma) = 1$). If we define the functions $f_2(x)$, $f_3(x)$ ($x \in \{0,1\}$) by

$$f_2(0) = 1, \quad f_2(1) = 3, \quad f_3(0) = 1, \quad f_3(1) = 2, \tag{57}$$

and the function $v(\Sigma)$ by

$$v(\Sigma) = f_2(X_3(\Sigma)), \tag{58}$$

the base vertex of the domain $S(\Sigma^+)$ will be the vertex $V(\tau_{v(\Sigma)}(\Sigma))$. The direction of the left edge of the domain $S(\Sigma^+)$ is then equal to $t(X_3(\Sigma), O(\Sigma))$, where the function $t(x, o)$ ($x \in \{0, 1\}$, $o \in \{0, 1\}$) is given by

$$t(0, 0) = 2, \quad t(0, 1) = 3, \quad t(1, 0) = 3, \quad t(1, 1) = 2, \tag{59}$$

and we finally obtain

$$\tau_1(\Sigma e(O(\Sigma), 3)) = \tau_3(\Sigma 0) = \tau_3(\Sigma 1) = \tau_{v(\Sigma)}(\Sigma) t(X_3(\Sigma), O(\Sigma)). \tag{60}$$

If the base vertex of the domain $S(\Sigma)$ is the vertex $V(\mathrm{T})$, then

$$\mathrm{T} = \tau_2(\Sigma); \tag{61}$$

comparing the formulae (38) – (43) and (51) – (54) we easily obtain that for any vertex $V(\mathrm{T})$ (which is not a polar one) and any $o \in \{0, 1\}$, it is always $\mathrm{T} = \tau_2(\sigma(o, \mathrm{T}))$.

We can now extend the formulae (1), (2) for the radius-vectors of all vertices of the net

$$\boldsymbol{e}(\mathrm{T}) = \boldsymbol{v}(V(\mathrm{T})) \tag{62}$$

and we shall investigate the properties of vertices of the degree 1. Using the formulae (34), (36) – (39), (23) and (16) – (19) we obtain the following vertices which are the centres of 30 edges of icosahedron:

$$V(a01) = C_1(a00) = C(V(a0), V(a^+0)), \tag{63}$$

$$V(a02) = C_2(a01) = C(V(a0), V(a1)), \tag{64}$$

$$V(a03) = C_3(a00) = C(V(a0), V(00)), \tag{65}$$

$$V(a11) = C_1(a11) = C(V(a1), V(a^+1)), \tag{66}$$

$$V(a12) = C_2(a10) = C(V(a1), V(01)), \tag{67}$$

$$V(a13) = C_3(a11) = C(V(a1), V(a^+0)). \tag{68}$$

Then we obtain using the formulae (5), (14), (15), (20), (21) and (62) the radius-vectors

$$\boldsymbol{e}(a03) = (\sin\xi_{10}\,\cos{(2a-2)\psi_5},\, \sin\xi_{10}\,\sin{(2a-2)\psi_5},\, \cos\xi_{10}), \tag{69}$$

$$\boldsymbol{e}(a01) = (\cos\xi_{10}\,\cos{(2a-1)\psi_5},\, \cos\xi_{10}\,\sin{(2a-1)\psi_5},\, \sin\xi_{10}), \tag{70}$$

$$\boldsymbol{e}(a02) = (\cos{(2a-3/2)\psi_5},\, \sin{(2a-3/2)\psi_5},\, 0), \tag{71}$$

$$\boldsymbol{e}(a13) = (\cos{(2a-1/2)\psi_5},\, \sin{(2a-1/2)\psi_5},\, 0), \tag{72}$$

$$\boldsymbol{e}(a11) = (\cos\xi_{10}\,\cos 2a\psi_5,\, \cos\xi_{10}\,\sin 2a\psi_5,\, -\sin\xi_{10}), \tag{73}$$

$$\boldsymbol{e}(a12) = (\sin\xi_{10}\,\cos{(2a-1)\psi_5},\, \sin\xi_{10}\,\sin{(2a-1)\psi_5},\, -\cos\xi_{10}), \tag{74}$$

where

$$\xi_{10} = \frac{1}{2}\,\xi_5; \tag{75}$$

according to (4) and (13)

$$\cos\xi_{10} = \sqrt{\frac{5+\sqrt{5}}{10}}, \quad \sin\xi_{10} = \sqrt{\frac{5-\sqrt{5}}{10}}, \tag{76}$$

and from (12) we obtain $\cos\xi_{10} = 2\cos\psi_5\sin\xi_{10}$.

Using the expressions for the radius-vectors (5), (14), (15), (69) – (74) and the definition of a hemisphere with the given pole (see Section 4), we can easily prove the following:
– each centre of an edge of a domain of degree 0 is a pole of a hemisphere whose boundary contains two edges of domains of degree 0;
– each vertex of degree 0 is a pole of a hemisphere whose boundary contains ten edges of domains of degree 1 (none from these edges is a part of an edge of domain of degree 0).
Thus there are 15 great circles dividing hemispheres whose poles are centres of edges of domains of degree 0 and 6 great circles dividing hemispheres whose poles are vertices of degree 0.

We present below the vertices of degree 0 and 1 which lie at these great circles (for $a \in \{1,2,3,4,5\}$): we list first the poles of the circle and then the vertices lying at the circle; the order of vertices at each circle is such that the northern pole of the circle is the first pole. The vertices of each circle are written in two rows; the two vertices in each column are mutually antipodal.

Poles:	Vertices:				
$V(a^-03)$	$V(a^{--}1)$	$V(a^{--}11)$	$V(a^-1)$	$V(a02)$	
$V(a^+12)$	$V(a^+0)$	$V(a^+01)$	$V(a^{++}0)$	$V(a^{++}13)$	
$V(a^-01)$	$V(a^{--}0)$	$V(a^{--}02)$	$V(a^{--}1)$	$V(a^-12)$	
$V(a^+11)$	$V(a1)$	$V(a13)$	$V(a^+0)$	$V(a^{++}03)$	
$V(a^-02)$	$V(a^{++}11)$	$V(01)$	$V(a12)$	$V(a1)$	
$V(a^+13)$	$V(a01)$	$V(00)$	$V(a^{--}03)$	$V(a^{--}0)$	
$V(00)$	$V(102)$	$V(113)$	$V(202)$	$V(213)$	$V(302)$
$V(01)$	$V(313)$	$V(402)$	$V(413)$	$V(502)$	$V(513)$
$V(a^-0)$	$V(a^{--}12)$	$V(a^-12)$	$V(a^-11)$	$V(a02)$	$V(a01)$
$V(a^+1)$	$V(a^+03)$	$V(a^{++}03)$	$V(a^{++}01)$	$V(a^{++}13)$	$V(a^{++}11)$

6. Properties of domains

According to the algorithm from the Section 4 we can construct arbitrarily small triangular domains at the unit sphere. On the contrary to the original domains $S(apq)$, which are mutually equal, in general the smaller domains need not be similar. Here we derive the constraints for the shape and area of these domains.

Consider the triangular domains $S(\Sigma)$ at the unit sphere; in the first part of this Section we shall ignore the definitions of the radius-vectors of vertices of these domains as presented in the previous Sections. We shall retain only the denotation of domains, their vertices and centres of their edges (formulae (23) and (24) – (27)). For any domain $S(\Sigma)$ we shall only require that the lengths of its edges are smaller than $\pi/2$ and that its internal angles are smaller than π (the last requirement is evidently true for any domain $S(\Sigma)$).

In the next we shall always assume that for any formula containing at least two from among the indices i, j, k, these indices belong to the ordered triple (i,j,k). Let $\lambda_i(\Sigma)$ be the length of the i-th edge of the domain $S(\Sigma)$: these lengths thus satisfy the condition

$$i \in \{1,2,3\}: \quad 0 < \lambda_i(\Sigma) < \frac{\pi}{2}; \tag{77}$$

if we define for brevity for each $i \in \{1,2,3\}$

$$c_i(\Sigma) = \cos\lambda_i(\Sigma), \quad s_i(\Sigma) = \sin\lambda_i(\Sigma), \tag{78}$$

we evidently have

$$c_i(\Sigma) = \boldsymbol{v}(V_j(\Sigma))\cdot\boldsymbol{v}(V_k(\Sigma)). \tag{79}$$

We further define according to (22) for each $i \in \{1,2,3\}$ the vector normal to the i-th edge of the domain $S(\Sigma)$

$$\boldsymbol{w}_i(\Sigma) = \boldsymbol{w}(V_j(\Sigma), V_k(\Sigma)); \tag{80}$$

these vectors are defined as the lenghts of the edges satisfy the condition (77). Let $\alpha_i(\Sigma)$ be the internal angle of the domain $S(\Sigma)$ by its i-th vertex; then

$$\cos\alpha_i(\Sigma) = -\,\boldsymbol{w}_j(\Sigma)\cdot\boldsymbol{w}_k(\Sigma), \tag{81}$$

$$\sin\alpha_i(\Sigma) = |\boldsymbol{w}_j(\Sigma)\times\boldsymbol{w}_k(\Sigma)|. \tag{82}$$

From (22), (79) and (78) we easily obtain the formulae

$$\cos\alpha_i(\Sigma) = \frac{c_i(\Sigma) - c_j(\Sigma)c_k(\Sigma)}{s_j(\Sigma)s_k(\Sigma)}, \tag{83}$$

$$\sin\alpha_i(\Sigma) = \frac{|[\boldsymbol{v}(V_i(\Sigma)), \boldsymbol{v}(V_j(\Sigma)), \boldsymbol{v}(V_k(\Sigma))]|}{s_j(\Sigma)s_k(\Sigma)}, \tag{84}$$

where for any vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = \boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) \tag{85}$$

is the triple product of these vectors. As the triple product on the r.h.s. of the formula (84) is the same for any ordered triple (i,j,k), we can denote

$$T(\Sigma) = [\boldsymbol{v}(V_i(\Sigma)), \boldsymbol{v}(V_j(\Sigma)), \boldsymbol{v}(V_k(\Sigma))]. \tag{86}$$

As the vertices of the domain $S(\Sigma)$ are ordered in the positive sense and their radius-vectors do not lie in a single plane, we have always

$$T(\Sigma) > 0. \tag{87}$$

From (83) and (84) we then obtain

$$(c_i(\Sigma) - c_j(\Sigma)c_k(\Sigma))^2 + T(\Sigma)^2 = s_j(\Sigma)^2 s_k(\Sigma)^2$$

and

$$T(\Sigma)^2 = 1 - c_1(\Sigma)^2 - c_2(\Sigma)^2 - c_3(\Sigma)^2 + 2\, c_1(\Sigma)c_2(\Sigma)c_3(\Sigma). \tag{88}$$

For any domain $S(\Sigma)$ and any ordered triple (i,j,k), the vector

$$(\boldsymbol{v}(V_i(\Sigma))-\boldsymbol{v}(V_k(\Sigma)))\times(\boldsymbol{v}(V_j(\Sigma))-\boldsymbol{v}(V_k(\Sigma)))$$

has the absolute value equal to twice the area of the (planar) triangle defined by the vertices of the domain $S(\Sigma)$ and it has the direction of the external normal to the plane of this triangle (it points away from the centre of the unit sphere). If we denote this vector as $\boldsymbol{s}(\Sigma)$, we get the expression

$$\begin{aligned}\boldsymbol{s}(\Sigma) = {} & \boldsymbol{v}(V_1(\Sigma))\times\boldsymbol{v}(V_2(\Sigma))+\boldsymbol{v}(V_2(\Sigma))\times\boldsymbol{v}(V_3(\Sigma))+ \\ & +\boldsymbol{v}(V_3(\Sigma))\times\boldsymbol{v}(V_1(\Sigma))\end{aligned} \tag{89}$$

and from (80), (22) and (78) we obtain

$$\boldsymbol{s}(\Sigma) = s_1(\Sigma)\boldsymbol{w}_1(\Sigma)+s_2(\Sigma)\boldsymbol{w}_2(\Sigma)+s_3(\Sigma)\boldsymbol{w}_3(\Sigma). \tag{90}$$

These formulae are not very suitable if the size of the domain $S(\Sigma)$ is very small. It can be easily shown that the vector $\boldsymbol{s}(\Sigma)$ can be written in the form

$$\boldsymbol{s}(\Sigma) = a_1(\Sigma)\boldsymbol{v}(V_1(\Sigma))+a_2(\Sigma)\boldsymbol{v}(V_2(\Sigma))+a_3(\Sigma)\boldsymbol{v}(V_3(\Sigma)), \tag{91}$$

where the coefficients $a_i(\Sigma)$ ($i \in \{1,2,3\}$) can be obtained by scalar multiplying of the r.h.s. of (91) and (90) by the vectors $\boldsymbol{w}_i(\Sigma)$ and using the formulae (81), (83), (80), (22), (85) and (86):

$$a_i(\Sigma) = \frac{(1-c_i(\Sigma))(1+c_i(\Sigma)-c_j(\Sigma)-c_k(\Sigma))}{T(\Sigma)}. \tag{92}$$

The square of the vector $\boldsymbol{s}(\Sigma)$ is according to (90) and (81) given by

$$|\boldsymbol{s}(\Sigma)|^2 = s_1(\Sigma)^2 + s_2(\Sigma)^2 + s_3(\Sigma)^2 - 2\, s_1(\Sigma)s_2(\Sigma)\cos\alpha_3(\Sigma) - \\ -2\, s_2(\Sigma)s_3(\Sigma)\cos\alpha_1(\Sigma) - 2\, s_3(\Sigma)s_1(\Sigma)\cos\alpha_2(\Sigma) \tag{93}$$

and the unit vector

$$\boldsymbol{n}(\Sigma) = \frac{\boldsymbol{s}(\Sigma)}{|\boldsymbol{s}(\Sigma)|} \tag{94}$$

is the radius-vector of the point of the unit sphere which is the projection (from the centre of the unit sphere) of the centre of the circle circumscribed to the triangle defined by the vertices of the domain $S(\Sigma)$. Therefore the point with the radius-vector $\boldsymbol{n}(\Sigma)$ can be considered as the centre of the triangular domain $S(\Sigma)$.

Consider now the domains $S(\Sigma r)$ ($r \in \{0, 1, 2, 3\}$) whose vertices are defined by the formulae (23) and (24) – (27). Using the formulae (20) and (21) we obtain

$$\boldsymbol{v}(C_j(\Sigma))\cdot\boldsymbol{v}(C_k(\Sigma)) = c_i^\circ(\Sigma) = \frac{A(\Sigma)}{4\, c_j^*(\Sigma)c_k^*(\Sigma)}, \tag{95}$$

$$\boldsymbol{v}(C_i(\Sigma))\cdot\boldsymbol{v}(V_j(\Sigma)) = \boldsymbol{v}(C_i(\Sigma))\cdot\boldsymbol{v}(V_k(\Sigma)) = c_i^*(\Sigma), \tag{96}$$

where (for all $i \in \{1, 2, 3\}$)

$$c_i^*(\Sigma) = \cos\lambda_i^*(\Sigma), \quad c_i^\circ(\Sigma) = \cos\lambda_i^\circ(\Sigma), \tag{97}$$

$$\lambda_i^*(\Sigma) = \frac{\lambda_i(\Sigma)}{2}, \tag{98}$$

and

$$A(\Sigma) = 1 + c_1(\Sigma) + c_2(\Sigma) + c_3(\Sigma). \tag{99}$$

If we define

$$K(\Sigma) = \frac{A(\Sigma)}{4\, c_1^*(\Sigma)c_2^*(\Sigma)c_3^*(\Sigma)}, \tag{100}$$

we obtain from (95) for all $i \in \{1, 2, 3\}$

$$c_i^\circ(\Sigma) = K(\Sigma) c_i^*(\Sigma). \tag{101}$$

Using (97), (98), (99) and (88) we can easily calculate that

$$(4\, c_1^*(\Sigma) c_2^*(\Sigma) c_3^*(\Sigma))^2 = 2\,(1+c_1(\Sigma))(1+c_2(\Sigma))(1+c_3(\Sigma)) = A(\Sigma)^2 + T(\Sigma)^2.$$

According to (77) we have $0 < c_i(\Sigma) < 1$ (and thus also $1/\sqrt{2} < c_i^*(\Sigma) < 1$) for all $i \in \{1,2,3\}$, what implies that $1 < A(\Sigma) < 4$ and

$$K(\Sigma) = \frac{A(\Sigma)}{\sqrt{A(\Sigma)^2 + T(\Sigma)^2}}. \tag{102}$$

From the inequality (87) we then obtain

$$0 < K(\Sigma) < 1, \tag{103}$$

and the formula (101) implies that for all $i \in \{1,2,3\}$

$$\lambda_i^\circ(\Sigma) > \lambda_i^*(\Sigma). \tag{104}$$

From (24) – (27) we obtain the following equalities for the edges of domains $S(\Sigma r)$:

$$\lambda_1(\Sigma 0) = \lambda_1^\circ(\Sigma), \quad \lambda_2(\Sigma 0) = \lambda_2^\circ(\Sigma), \quad \lambda_3(\Sigma 0) = \lambda_3^\circ(\Sigma), \tag{105}$$

$$\lambda_1(\Sigma 1) = \lambda_1^\circ(\Sigma), \quad \lambda_2(\Sigma 1) = \lambda_2^*(\Sigma), \quad \lambda_3(\Sigma 1) = \lambda_3^*(\Sigma), \tag{106}$$

$$\lambda_1(\Sigma 2) = \lambda_1^*(\Sigma), \quad \lambda_2(\Sigma 2) = \lambda_2^\circ(\Sigma), \quad \lambda_3(\Sigma 2) = \lambda_3^*(\Sigma), \tag{107}$$

$$\lambda_1(\Sigma 3) = \lambda_1^*(\Sigma), \quad \lambda_2(\Sigma 3) = \lambda_2^*(\Sigma), \quad \lambda_3(\Sigma 3) = \lambda_3^\circ(\Sigma). \tag{108}$$

Then we can calculate the quantity $T(\Sigma r)$ and the internal angles for these domains by inserting the presented lengths of edges instead of the original ones in the formulae (88), (83) and (84). Using the formulae (78), (97) – (102) and (87) we easily obtain

$$T(\Sigma 0) = \frac{T(\Sigma)}{\sqrt{A(\Sigma)^2 + T(\Sigma)^2}} = \frac{T(\Sigma)}{4\, c_1^*(\Sigma) c_2^*(\Sigma) c_3^*(\Sigma)} \tag{109}$$

and for $i \in \{1,2,3\}$

$$T(\Sigma i) = \frac{T(\Sigma)\, c_i^*(\Sigma)}{\sqrt{A(\Sigma)^2 + T(\Sigma)^2}} = \frac{T(\Sigma)}{4\, c_j^*(\Sigma) c_k^*(\Sigma)}. \tag{110}$$

According to (77) we have $\sqrt{2} < 4c_1^*(\Sigma)c_2^*(\Sigma)c_3^*(\Sigma) < 4$; therefore the quantities $T(\Sigma r)$ are always smaller than $T(\Sigma)$.

Let us now assume that the lengths of edges $\lambda_i(\Sigma)$ satisfy the inequality

$$i \in \{1,2,3\}: \quad 0 < \lambda^-(\Sigma) \le \lambda_i(\Sigma) \le \lambda^+(\Sigma) < \frac{\pi}{2}, \tag{111}$$

where $\lambda^-(\Sigma)$, $\lambda^+(\Sigma)$ are some parameters. Then we obtain from (98) the bound for the quantities $\lambda_i^*(\Sigma)$

$$i \in \{1,2,3\}: \quad \frac{\lambda^-(\Sigma)}{2} \le \lambda_i^*(\Sigma) \le \frac{\lambda^+(\Sigma)}{2}. \tag{112}$$

On the other hand, using the Cauchy inequality we obtain

$$(4c_j^*(\Sigma)c_k^*(\Sigma))^2 = 4\,(1{+}c_j(\Sigma))(1{+}c_k(\Sigma)) \le (2{+}c_j(\Sigma){+}c_k(\Sigma))^2,$$

and thus

$$\frac{A(\Sigma)}{4c_j^*(\Sigma)c_k^*(\Sigma)} \ge \frac{A(\Sigma)}{2{+}c_j(\Sigma){+}c_k(\Sigma)} = 1 - \frac{1 - c_i(\Sigma)}{2{+}c_j(\Sigma){+}c_k(\Sigma)}.$$

The quantity on the r.h.s. is according to (78) and (77) a decreasing function of all its parameters $\lambda_1(\Sigma)$, $\lambda_2(\Sigma)$, $\lambda_3(\Sigma)$, and thus according to (111) it is greater than or equal to its value for $\lambda_1(\Sigma) = \lambda_2(\Sigma) = \lambda_3(\Sigma) = \lambda^+(\Sigma)$:

$$1 - \frac{1 - c_i(\Sigma)}{2{+}c_j(\Sigma){+}c_k(\Sigma)} \ge 1 - \frac{1 - \cos\lambda^+(\Sigma)}{2(1 + \cos\lambda^+(\Sigma))}.$$

If we define

$$b(u) = 1 - \frac{1-u}{2(1{+}u)}, \tag{113}$$

using the formula (95) we get

$$c_i^\circ(\Sigma) \ge b(\cos\lambda^+(\Sigma)) \tag{114}$$

and using (104) and (112) we obtain the bound

$$i \in \{1,2,3\}: \quad \frac{\lambda^-(\Sigma)}{2} < \lambda_i^\circ(\Sigma) \le \beta(\lambda^+(\Sigma)), \tag{115}$$

where the fuction $\beta(\lambda)$ is defined for $0 \le \lambda \le \pi/2$ as

$$\beta(\lambda) = \arccos\, b(\cos\lambda). \tag{116}$$

In order to see explicitly the behaviour of the function $b(u)$, we calculate the bounds for this function in the interval $0 \le u \le 1$. We have

$$b(u) - u = (1-u)\,\frac{1+2u}{2(1+u)};$$

if we denote for brevity

$$r(u) = \sqrt{\frac{1+u}{2}}, \tag{117}$$

we have

$$1 - r(u) = \frac{1-u}{2(1+r(u))}, \quad r(u) - u = \frac{(1-u)(1+2u)}{2(u+r(u))},$$

and after an easy calculation we obtain

$$r(u) - b(u) = \frac{1-u}{2(1+u)} - \frac{1-u}{2(1+r(u))} = (1-u)^2\,\frac{1+2u}{4(1+u)(1+r(u))(u+r(u))}.$$

For $0 < u < 1$ we get the inequality

$$u < b(u) < r(u) \tag{118}$$

and thus (as $\cos\lambda/2 = r(\cos\lambda)$ for $0 \le \lambda \le \pi$)

$$\frac{\lambda^+(\Sigma)}{2} < \beta(\lambda^+(\Sigma)) < \lambda^+(\Sigma). \tag{119}$$

According to (105) – (108) we finally obtain from (112) and (115) for every $r \in \{0,1,2,3\}$ the bound

$$i \in \{1,2,3\}: \quad \frac{\lambda^-(\Sigma)}{2} \le \lambda_i(\Sigma r) \le \beta(\lambda^+(\Sigma)). \tag{120}$$

In the next we shall need an expression for the iterated function $b(u)$:

$$b_0(u) = u, \quad n \ge 0: \quad b_{n+1}(u) = b(b_n(u)) \tag{121}$$

(note that $b_1(u) = b(u)$). We first observe that the function $b(u)$ is a rational function of u whose nominator and denominator are linear polynomials of u and whose value for $u = 1$ is 1. If we calculate the function $b(b(u))$, we see that this is again a rational function of u of the same type. We therefore propose to write the function $b_n(u)$ in the form

$$b_n(u) = 1 - \frac{1-u}{a_n + b_n u}. \tag{122}$$

The equations for the unknown coefficients a_n, b_n we find by inserting this expression in the formula (121):

$$a_0 = 1, \quad b_0 = 0, \quad n \geq 0: \quad a_{n+1} = 4a_n - 2, \quad b_{n+1} = 4b_n + 2.$$

It is clear that both coefficients a_n and b_n have to be a linear combinations of 4^n and 1, and we easily obtain the expressions

$$a_n = \frac{4^n + 2}{3}, \quad b_n = \frac{2(4^n - 1)}{3}.$$

Inserting in the formula (122) we obtain for the function $b_n(u)$ for any $n \geq 0$ the formula

$$b_n(u) = 1 - \frac{3(1-u)}{4^n(1+2u) + 2(1-u)}. \tag{123}$$

Now we can derive from the formulae (111) and (120) the bound for the lengths of edges of any domain $S(\Sigma)$. Using the formulae (30) and (121) we obtain

$$i \in \{1,2,3\}: \quad \frac{\lambda^-(\Sigma_0)}{2^{N(\Sigma)}} \leq \lambda_i(\Sigma) \leq \beta_{N(\Sigma)}(\lambda^+(\Sigma_0)), \tag{124}$$

where we have defined in analogy with (116) for $0 \leq \lambda \leq \pi/2$

$$\beta_n(\lambda) = \arccos b_n(\cos\lambda). \tag{125}$$

At this point we apply all properties of domains $S(\Sigma)$ as presented in the previous Sections. This means according to (5), (14), (15) and (79) that for all $i \in \{1,2,3\}$ we have $c_i(\Sigma_0) = c$, where c is given by (13). Using the formula (78) we get that the length of the i-th edge of any domain $S(\Sigma_0)$ is

$$\lambda_i(\Sigma_0) = \xi_5, \tag{126}$$

where ξ_5 is defined by (4), and thus the condition (77) is evidently satisfied. Therefore we can put

$$\lambda^-(\Sigma_0) = \lambda^+(\Sigma_0) = \xi_5, \tag{127}$$

and inserting in the formula (124) we finally obtain

$$i \in \{1,2,3\}: \quad \frac{\xi_5}{2^{N(\Sigma)}} \le \lambda_i(\Sigma) \le \beta_{N(\Sigma)}(\xi_5). \tag{128}$$

Now we examine the behaviour of this bound for decreasing size of the domain $S(\Sigma)$. According to (123) the function $b_n(u)$ increases with increasing n for any $0 < u < 1$; therefore, according to (125) the function $\beta_n(\lambda)$ decreases with increasing n for any $0 < \lambda < \pi/2$. For $0 < u < 1$ we have according to (123) $0 < b_n(u) < 1$; from (118) we then get the inequality $b(b_n(u)) < r(b_n(u))$ and according to (121) we obtain $b_{n+1}(u) < r(b_n(u))$. Using (117) and (125) we derive for $0 < \lambda < \pi/2$ the inequality $\beta_{n+1}(\lambda) > \beta_n(\lambda)/2$ and this implies that $2^n\beta_n(\lambda)$ is an increasing function of n. As $\arccos u = \arcsin\sqrt{1-u^2}$ for any $0 \le u \le 1$, using the expression (123) we easily calculate

$$\lim_{n\to\infty} 2^n\beta_n(\lambda) = \lim_{n\to\infty} 2^n \arcsin\sqrt{(1-b_n(\cos\lambda))(1+b_n(\cos\lambda))} =$$
$$= \lim_{n\to\infty} 2^n\sqrt{2\,(1-b_n(\cos\lambda))} = \sqrt{\frac{6(1-\cos\lambda)}{1+2\cos\lambda}}.$$

Inserting ξ_5 for λ and using the formulae (4) and (13) we obtain

$$\lim_{n\to\infty} 2^n\beta_n(\xi_5) = \sqrt{\frac{6(1-\cos\xi_5)}{1+2\cos\xi_5}} = \sqrt{6(7-3\sqrt{5})}. \tag{129}$$

Numerical value of this limit (with the precision of 10 decimal digits) is 1.3231690765, while the value of ξ_5 is 1.1071487178 (the ratio of these two numbers is 1.1951141299).

It can be easily shown that both the upper and the lower bound in the formula (128) are actually reached for some edges: from (105) – (108) we obtain that the value of the lower bound is acquired by any edge which is a part of some edge of an original domain $S(\Sigma_0)$ (thus for the i-th edge of the domain $S(\Sigma_0\Sigma_{\{j,k\}})$ where (i,j,k) is an ordered triple); the value of the upper bound is acquired by any edge of any domain $S(\Sigma_0\Sigma_{\{0\}})$ (and also for the i-th edge of the domain $S(\Sigma_0 i)$). However, there do not exist arbitrarily small domains whose one edge has the minimal and another edge the maximal length: this is true only for the domains $S(\Sigma_0)$ and $S(\Sigma_0 i)$ (where $i \in \{1,2,3\}$).

The bound (128) for the length of the edges of domains $S(\Sigma)$ represents also the bound on the internal angles of these domains: according to the formulae (84), (86) and (87) we have

$$\sin\alpha_i(\Sigma) = \frac{T(\Sigma)}{s_j(\Sigma)s_k(\Sigma)} = \frac{T(\Sigma)\,s_i(\Sigma)}{s_1(\Sigma)s_2(\Sigma)s_3(\Sigma)}. \tag{130}$$

Therefore the ratio of the sines $\sin\alpha_i(\Sigma)$, $\sin\alpha_j(\Sigma)$ is equal to the ratio of the sines $s_i(\Sigma)$, $s_j(\Sigma)$ (the well-known sine law for spherical triangles) and the latter ratio is bounded according to (78) and (128).

7. Position of a point

After we have defined the domains which are parts of the spherical surface, it is necessary to construct an algorithm, which allows to determine for any point of the unit sphere the domain (of certain size, thus of certain degree) containing this point. As each domain is an open set, it may happen that the given point lies at the boundary of two domains (thus at the common edge of these domains) or even at the boundary of more than two domains (thus at the common vertex of these domains).

Consider any fixed domain $S(\Sigma)$; the vectors $\boldsymbol{w}_i(\Sigma)$ of internal normals to the edges of this domain are defined by (80). The domain $S(\Sigma)$ is the intersection of the hemispheres $H(\boldsymbol{w}_i(\Sigma))$ ($i \in \{1,2,3\}$); therefore any point P of the unit sphere lies in the domain $S(\Sigma)$ just if it lies in each of these hemispheres. We define for each $i \in \{1,2,3\}$ the quantity

$$h_i(P,\Sigma) = \boldsymbol{v}(P)\cdot\boldsymbol{w}_i(\Sigma), \tag{131}$$

which will be called the height of the point P (with respect to the i-th edge of the domain $S(\Sigma)$): it is the height of this point with respect to the plane of this edge with positive values for the points of the hemisphere $H(\boldsymbol{w}_i(\Sigma))$. Then the point P of the unit sphere lies in the domain $S(\Sigma)$ just if

$$i \in \{1,2,3\}: \quad h_i(P,\Sigma) > 0; \tag{132}$$

this point lies in this domain or at its boundary just if

$$i \in \{1,2,3\}: \quad h_i(P,\Sigma) \geq 0. \tag{133}$$

Note, that it is not possible that $h_i(P,\Sigma) = 0$ for all $i \in \{1,2,3\}$. Let (i,j,k) be an ordered triple; if $h_i(P,\Sigma) = 0$, $h_j(P,\Sigma) = 0$, $h_k(P,\Sigma) > 0$, the point P is identical with the vertex $V_k(\Sigma)$; if $h_i(P,\Sigma) > 0$, $h_j(P,\Sigma) > 0$,

$h_k(P,\Sigma) = 0$, the point P lies at the k-th edge of the domain $S(\Sigma)$ between the vertices $V_i(\Sigma)$ and $V_j(\Sigma)$.

If we divide the domain $S(\Sigma)$ into domains $S(\Sigma r)$ ($r \in \{0,1,2,3\}$), we need not to calculate 12 new values of heights according to (131), as there are only 3 new heights $h_i(P,\Sigma 0)$. The other ones can be obtained according to the formulae (24) – (27):

$$h_1(P,\Sigma 1) = -h_1(P,\Sigma 0), \quad h_2(P,\Sigma 1) = h_2(P,\Sigma), \quad h_3(P,\Sigma 1) = h_3(P,\Sigma), \tag{134}$$

$$h_1(P,\Sigma 2) = h_1(P,\Sigma), \quad h_2(P,\Sigma 2) = -h_2(P,\Sigma 0), \quad h_3(P,\Sigma 2) = h_3(P,\Sigma), \tag{135}$$

$$h_1(P,\Sigma 3) = h_1(P,\Sigma), \quad h_2(P,\Sigma 3) = h_2(P,\Sigma), \quad h_3(P,\Sigma 3) = -h_3(P,\Sigma 0). \tag{136}$$

Thus the algorithm for determination of domains which contain the given point P may look as follows: it has an initial step and some number of iterative steps. In each step we consider domains $S(\Sigma)$ with some fixed degree N: in the initial step we have $N = 0$, while in the n-th iterative step $N = n$.

Initial step: for each $a \in \{1,2,3,4,5\}$, $p \in \{0,1\}$, $q \in \{0,1\}$ and $i \in \{1,2,3\}$ we calculate according to (131) the heights $h_i(P,apq)$ (totally 60 heights). In fact we need to calculate only 15 heights, as each edge belongs to the boundary of two neighbouring domains and each two antipodal edges lie at the same great circle (see the table at the end of Section 5). For each of these 15 great circles we can choose one of its poles, for example, the first pole presented in the table; then we can easily obtain from (80), (22), (16) – (19), (62) and the expressions for the radius-vectors (5), (14), (15), (69) – (74) the following expressions for the vectors $\boldsymbol{w}_i(apq)$:

$$\boldsymbol{w}_1(a00) = \boldsymbol{e}(a^{--}03), \ \ \boldsymbol{w}_2(a00) = -\boldsymbol{e}(a^{++}02), \ \ \boldsymbol{w}_3(a00) = \boldsymbol{e}(a^{+}02), \tag{137}$$

$$\boldsymbol{w}_1(a01) = -\boldsymbol{e}(a^{--}03), \ \ \boldsymbol{w}_2(a01) = \boldsymbol{e}(a^{+}01), \ \ \boldsymbol{w}_3(a01) = \boldsymbol{e}(a^{-}01), \tag{138}$$

$$\boldsymbol{w}_1(a10) = -\boldsymbol{e}(a^{+}03), \ \ \boldsymbol{w}_2(a10) = -\boldsymbol{e}(a^{-}02), \ \ \boldsymbol{w}_3(a10) = \boldsymbol{e}(a02), \tag{139}$$

$$\boldsymbol{w}_1(a11) = \boldsymbol{e}(a^{+}03), \quad \boldsymbol{w}_2(a11) = -\boldsymbol{e}(a^{++}01), \quad \boldsymbol{w}_3(a11) = -\boldsymbol{e}(a^{-}01). \quad (140)$$

Then we can determine those domains $S(\Sigma)$ which satisfy the condition (133). There are three possible cases:

Case 1: there is exactly one such domain. This means that the point P lies in this domain.

Case 2: there are exactly two such domains. This means that the point P lies at the common edge of these two domains.

Case 3: there are exactly five such domains. This means that the point P lies at the common vertex of these five domains.

In the case 3 the determination is finished, as it is possible to find for any $N' \geq 0$ all 5 domains $S(\Sigma')$ of degree N' whose common vertex is the point P. In the other two cases the determination continues by the next step with a single domain (case 1) or with two neighbouring domains (case 2).

Iterative step: for each Σ determined in the previous step, for each $r \in \{0,1,2,3\}$ and $i \in \{1,2,3\}$ we calculate according to (131) the heights $h_i(P,\Sigma r)$ (totally 12 heights for each domain from the previous step). As mentioned above, we need to calculate only the heights $h_i(P,\Sigma 0)$, while the other ones are given by (134) – (136).

Then we determine those domains $S(\Sigma r)$ which satisfy the condition (133). There are again three possible cases:

Case 1: there is exactly one such domain. This means that the point P lies in this domain.

Case 2: there are exactly two such domains. This means that the point P lies at the common edge of these two domains.

Case 3: there are exactly six such domains. This means that the point P lies at the common vertex of these six domains.

Note that the case 1 is possible only if this step has begun with a single domain, while the case 3 is possible only if this step has begun with two neighbouring domains.

In the case 3 the determination is finished, as it is possible to find for any $N' \geq N$ all 6 domains $S(\Sigma')$ of degree N' whose common vertex is the point P. In the other two cases the determination continues by the next step with a single domain (case 1) or with two neighbouring domains (case 2).

The determination ends if N acquires some predetermined value (which corresponds to the required precision of determination).

The inverse procedure – determination of the position of the domain $S(\Sigma)$ on the unit sphere is straightforward. For any $0 \leq n \leq N(\Sigma)$, let Σ_n be the initial segment of the sequence Σ consisting of the first $n+3$ digits (thus $\Sigma_{N(\Sigma)} = \Sigma$). For each $0 \leq n \leq N(\Sigma)$ we determine successively the radius-vectors of vertices of domains $S(\Sigma_n)$ (starting from the domain $S(\Sigma_0)$) using the formulae (16) – (19), (1), (2), (5), (14), (15), (20), (21), (23) and (24) – (27). After we have obtained in this way the radius-vectors of vertices of domain $S(\Sigma)$, we may calculate the radius-vector of the centre of domain $S(\Sigma)$ using the formula (94) and one of the formulae (89) – (91).

Another important task is the determination of the neighbourhood of the given point: this neighbourhood can be defined as the minimal set of domains of some degree such that any point of the unit sphere whose (angular or cartesian) distance from the given point is smaller than some given value, belongs to this set of domains. According to the results of Section 6, this task may be rather complicated because of the variable shape and size of the particular domains. Therefore it will be advantageous to abandon the requirement of minimality of this set of domains and to construct this set successively (starting from the domain(s) the given point belongs to) until we obtain a set surely containing all points whose distance from the given point is smaller than the given value.

The mentioned construction of a larger set of domains from the given one can be performed by the determination of the neighbouring domains $S(\Sigma^*)$, $S(\Sigma^-)$, $S(\Sigma^+)$ of the given domain $S(\Sigma)$ (see the Section 5 after the formula (45)). According to the definitions from the Section 3 (see also Fig. 1) we immediately obtain

$$(a00)^* = a01, \quad (a00)^- = a^-00, \quad (a00)^+ = a^+00, \tag{141}$$

$$(a01)^* = a00, \quad (a01)^- = a^-11, \quad (a01)^+ = a11, \tag{142}$$

$$(a10)^* = a11, \quad (a10)^- = a^-10, \quad (a10)^+ = a^+10, \tag{143}$$

$$(a11)^* = a10, \quad (a11)^- = a01, \quad (a11)^+ = a^+01. \tag{144}$$

According to the definitions from the Section 5, especially the formulae (31), (32) and (57) (see also Fig. 2), we get

$$(\Sigma 0)^* = \Sigma 1, \quad (\Sigma 1)^* = \Sigma 0, \quad (\Sigma 2)^* = \Sigma^* 3, \quad (\Sigma 3)^* = \Sigma^* 2, \tag{145}$$

$$(\Sigma 0)^- = \Sigma e(O(\Sigma), 2), \quad (\Sigma 0)^+ = \Sigma e(O(\Sigma), 3), \tag{146}$$

$$(\Sigma 1)^- = \Sigma^- e(O(\Sigma), f_3(X_2(\Sigma)^*)), \ (\Sigma 1)^+ = \Sigma^+ e(O(\Sigma), f_2(X_3(\Sigma)^*)), \tag{147}$$

$$(\Sigma e(O(\Sigma), 2))^- = \Sigma^- e(O(\Sigma), f_2(X_2(\Sigma))), \quad (\Sigma e(O(\Sigma), 2))^+ = \Sigma 0, \tag{148}$$

$$(\Sigma e(O(\Sigma), 3))^- = \Sigma 0, \quad (\Sigma e(O(\Sigma), 3))^+ = \Sigma^+ e(O(\Sigma), f_3(X_3(\Sigma))). \tag{149}$$

In order to facilitate the decision which domains have to be included in the constructed set of domains, we introduce the following definitions. As it is evident from the previous Sections, for any given vertex of the net of degree N there are 5 (if the minimal degree of this vertex is 0) or 6 (otherwise) edges of degree N whose one end point is the given vertex; the other end points of these edges are the neighbouring vertices of degree N of the given vertex. Now we define for any point P of the unit sphere which is a vertex of the net of degree N, the set $Q_{N,l}(P)$ of its neighbouring vertices of degree N and order l as follows: if $l = 0$, the set $Q_{N,0}(P)$ has the single member P; if $l > 0$, the members of the set $Q_{N,l}(P)$ are all members of the set $Q_{N,l-1}(P)$ and all neighbouring vertices of degree N of all members of the set $Q_{N,l-1}(P)$. Thus the members of the set $Q_{N,1}(P)$ are the vertex P itself and all its neighbouring vertices of degree N. Any domain $S(\Sigma)$ whose all vertices belong to the set $Q_{N,l}(P)$ will be called the neighbouring domain of degree N and order l of the vertex P. We define the neighbourhood of degree N and order l of the vertex P as the interior of the closure of all neighbouring domains of degree N and order l of the vertex P (thus this neighbourhood is an open set).

For any set $Q_{N,l}(P)$, the members of this set which are not members of the set $Q_{N,l-1}(P)$ will be called the boundary vertices of the set $Q_{N,l}(P)$. We shall require that all boundary vertices belong to the hemisphere $H(\boldsymbol{v}(P))$ (thus they are not too distant from the point P). Any neighbouring domain of degree N and order l of the vertex P whose two vertices are the boundary vertices of the set $Q_{N,l}(P)$ will be called the boundary domain of degree N and order l of the vertex P (it is clear that for any domain, at most two of its vertices can be the boundary ones).

Consider now all boundary domains of degree N and order l of the given vertex P; if $S(\Sigma)$ is such a domain and its vertices $V_j(\Sigma)$, $V_k(\Sigma)$ (where

(i,j,k) is an ordered triple) are the boundary vertices of the set $Q_{N,l}(P)$, we calculate using the formulae (80) and (131) the quantity $\lambda_{N,l}(P,\Sigma)$ defined by the formula $\cos\lambda_{N,l}(P,\Sigma) = h_i(P,\Sigma)$. Let $\lambda_{N,l}(P)$ be the minimal value of $\lambda_{N,l}(P,\Sigma)$ for all boundary domains $S(\Sigma)$ of degree N and order l of the vertex P; then it is clear that any point Q of the unit sphere whose angular distance from the vertex P is smaller than $\lambda_{N,l}(P)$ belongs to the neighbourhood of degree N and order l of the vertex P. This shows that it is easily possible to find for any N the value of l such that any point of the unit sphere whose angular distance from the given vertex P is smaller than some given value, belongs to the neighbourhood of degree N and order l of the vertex P.

The definition of the neighbourhood of degree N and order l can be generalized for any point P of the unit sphere: if this point is a vertex of degree N, we have the above definition, if this point lies on the edge connecting two vertices of degree N, its neighbourhood is the union of the neighbourhoods of degree N and order l of these two vertices, if this point lies in some domain of degree N, its neighbourhood is the union of the neighbourhoods of degree N and order l of all three vertices of this domain.

In order to find the neighbouring vertices of any vertex of the net, we introduce the operations $\alpha^+(\mathrm{T},d)$ and $\alpha^-(\mathrm{T},d)$ ($d \in \{1,2,3\}$) defined (if possible) for any vertex $V(\mathrm{T})$ which is not a polar one. Any such vertex is a base vertex of exactly three base edges (of the same degree as this vertex) with mutually different directions (see the Section 5 and Fig. 3). For the edge with direction d ($1 \le d \le 3$), the vertex $V(\mathrm{T})$ is the one end point of this edge; the other end point of this edge will be the vertex $V(\alpha^+(\mathrm{T},d))$.

Using the definitions from the Sections 2, 3 and 5 (see also Figs. 1 and 3) we obtain

$$\alpha^+(a0,1) = a^+0, \quad \alpha^+(a0,2) = a1, \quad \alpha^+(a0,3) = 00, \tag{150}$$

$$\alpha^+(a1,1) = a^+1, \quad \alpha^+(a1,2) = 01, \quad \alpha^+(a1,3) = a^+0. \tag{151}$$

The result of these operations for the vertices $V(\mathrm{T}s)$ ($s \in \{0,1,2,3\}$) depends according to the formulae (52) – (60) on the values of the functions $X_3(\sigma(0,\mathrm{T}))$ and $X_3(\sigma(1,\mathrm{T}))$ (see the formulae (34) and (46) – (49)). Using the formulae (38) – (43) we easily obtain that $X_3(\sigma(0,\mathrm{T})) = 1$ only if T has the form $a0\mathrm{T}_{\{1,3\}}$ and $X_3(\sigma(1,\mathrm{T})) = 1$ only if T has the form $a1\mathrm{T}_{\{1,2\}}$. Then we can easily derive the formulae

$$\alpha^+(\mathrm{T}0,1)=\mathrm{T}1,\quad \alpha^+(\mathrm{T}0,2)=\mathrm{T}2,\quad \alpha^+(\mathrm{T}0,3)=\mathrm{T}3, \tag{152}$$

$$\alpha^+(\mathrm{T}1,1)=\alpha^+(\mathrm{T},1)0,\quad \alpha^+(\mathrm{T}1,2)=g_2^+(\mathrm{T}),\quad \alpha^+(\mathrm{T}1,3)=g_3^+(\mathrm{T}), \tag{153}$$

$$\alpha^+(\mathrm{T}2,1)=g_2^+(\mathrm{T}),\quad \alpha^+(\mathrm{T}2,2)=\alpha^+(\mathrm{T},2)0,\quad \alpha^+(\mathrm{T}2,3)=\mathrm{T}1, \tag{154}$$

$$\alpha^+(\mathrm{T}3,1)=g_3^+(\mathrm{T}),\quad \alpha^+(\mathrm{T}3,2)=\mathrm{T}1,\quad \alpha^+(\mathrm{T}3,3)=\alpha^+(\mathrm{T},3)0, \tag{155}$$

where (see (57) and (59))

$$g_2^+(\mathrm{T})=\alpha^+(\mathrm{T},f_3(X_3(\sigma(1,\mathrm{T}))^*))t(X_3(\sigma(1,\mathrm{T})),1), \tag{156}$$

$$g_3^+(\mathrm{T})=\alpha^+(\mathrm{T},f_2(X_3(\sigma(0,\mathrm{T}))^*))t(X_3(\sigma(0,\mathrm{T})),0), \tag{157}$$

what can be explicitly written as

$$\begin{aligned} X_3(\sigma(1,\mathrm{T}))=0: &\quad g_2^+(\mathrm{T})=\alpha^+(\mathrm{T},2)3,\\ X_3(\sigma(1,\mathrm{T}))=1: &\quad g_2^+(\mathrm{T})=\alpha^+(\mathrm{T},1)2, \end{aligned} \tag{158}$$

$$\begin{aligned} X_3(\sigma(0,\mathrm{T}))=0: &\quad g_3^+(\mathrm{T})=\alpha^+(\mathrm{T},3)2,\\ X_3(\sigma(0,\mathrm{T}))=1: &\quad g_3^+(\mathrm{T})=\alpha^+(\mathrm{T},1)3. \end{aligned} \tag{159}$$

Consider now the vertices of some fixed degree with exception of the polar ones; for any such vertex $V(\mathrm{T})$ and for any $d\in\{1,2,3\}$ the operation $\alpha^+(\mathrm{T},d)$ is defined and its result is unique with exception of the following two anomalous cases:

$$X_3(\sigma(1,\mathrm{T}))=1:\quad \alpha^+(\mathrm{T}1,2)=\alpha^+(\alpha^+(\mathrm{T},1)0,2), \tag{160}$$

$$X_3(\sigma(0,\mathrm{T}))=1:\quad \alpha^+(\mathrm{T}1,3)=\alpha^+(\alpha^+(\mathrm{T},1)0,3). \tag{161}$$

We turn now to the operations $\alpha^-(\mathrm{T},d)$: we shall require that for any $d\in\{1,2,3\}$ the operation $\alpha^-(\mathrm{T},d)$ is (if possible) inverse to the operation $\alpha^+(\mathrm{T},d)$, thus the vertex $V(\mathrm{T})$ (which is not a polar one) should be the other end of the edge whose base vertex is $V(\alpha^-(\mathrm{T},d))$ and whose direction is d. On the contrary to the operations $\alpha^+(\mathrm{T},d)$, the operations $\alpha^-(\mathrm{T},d)$ are in some cases undefined, while in some other cases (see the formulae (160) and (161)) we have to decide between the two possible definitions of these operations. In the case of the vertices $V(ap)$ we have

$$\alpha^-(a0,1) = a^-0, \quad \alpha^-(a0,2) = \text{undef}, \quad \alpha^-(a0,3) = a^-1, \tag{162}$$

$$\alpha^-(a1,1) = a^-1, \quad \alpha^-(a1,2) = a0, \quad \alpha^-(a1,3) = \text{undef}, \tag{163}$$

where undef means that the operation is undefined. The result of these operations for the vertices $V(\mathrm{T}s)$ ($s \in \{0,1,2,3\}$) depends according to the formulae (52) – (60) on the values of the functions $X_2(\sigma(0,\mathrm{T}))$ and $X_2(\sigma(1,\mathrm{T}))$ (see the formulae (34) and (46) – (49)), as we evidently have $X_3(\Sigma^-) = X_2(\Sigma)$. Using the formulae (38) – (43) we easily obtain that $X_2(\sigma(0,\mathrm{T})) = 1$ only if T has the form $a0\mathrm{T}_{\{0,3\}}$ and $X_2(\sigma(1,\mathrm{T})) = 1$ only if T has the form $a1\mathrm{T}_{\{0,2\}}$. From the formulae (152) – (155) we then obtain after an easy calculation (in the cases where two definitions are possible we choose the simpler one)

$$\alpha^-(\mathrm{T}0,1) = \alpha^-(\mathrm{T},1)1, \quad \alpha^-(\mathrm{T}0,2) = \alpha^-(\mathrm{T},2)2, \quad \alpha^-(\mathrm{T}0,3) = \alpha^-(\mathrm{T},3)3, \tag{164}$$

$$\alpha^-(\mathrm{T}1,1) = \mathrm{T}0, \quad \alpha^-(\mathrm{T}1,2) = \mathrm{T}3, \quad \alpha^-(\mathrm{T}1,3) = \mathrm{T}2, \tag{165}$$

$$\alpha^-(\mathrm{T}2,1) = g_2^-(\mathrm{T}), \quad \alpha^-(\mathrm{T}2,2) = \mathrm{T}0, \quad \alpha^-(\mathrm{T}2,3) = \alpha^-(\mathrm{T},3)1, \tag{166}$$

$$\alpha^-(\mathrm{T}3,1) = g_3^-(\mathrm{T}), \quad \alpha^-(\mathrm{T}3,2) = \alpha^-(\mathrm{T},2)1, \quad \alpha^-(\mathrm{T}3,3) = \mathrm{T}0, \tag{167}$$

where

$$g_2^-(\mathrm{T}) = \alpha^-(\mathrm{T}, f_2(X_2(\sigma(1,\mathrm{T}))^*))t(X_2(\sigma(1,\mathrm{T})),1), \tag{168}$$

$$g_3^-(\mathrm{T}) = \alpha^-(\mathrm{T}, f_3(X_2(\sigma(0,\mathrm{T}))^*))t(X_2(\sigma(0,\mathrm{T})),0), \tag{169}$$

what can be explicitly written as

$$\begin{aligned} X_2(\sigma(1,\mathrm{T})) = 0: &\quad g_2^-(\mathrm{T}) = \alpha^-(\mathrm{T},3)3, \\ X_2(\sigma(1,\mathrm{T})) = 1: &\quad g_2^-(\mathrm{T}) = \alpha^-(\mathrm{T},1)2, \end{aligned} \tag{170}$$

$$\begin{aligned} X_2(\sigma(0,\mathrm{T})) = 0: &\quad g_3^-(\mathrm{T}) = \alpha^-(\mathrm{T},2)2, \\ X_2(\sigma(0,\mathrm{T})) = 1: &\quad g_3^-(\mathrm{T}) = \alpha^-(\mathrm{T},1)3. \end{aligned} \tag{171}$$

The formulae (160), (161) can be now written in the form

$$X_2(\sigma(1,\mathrm{T})) = 1: \quad \alpha^+(\alpha^-(\mathrm{T},1)1,2) = \mathrm{T}2, \tag{172}$$

$$X_2(\sigma(0,\mathrm{T})) = 1: \quad \alpha^+(\alpha^-(\mathrm{T},1)1,3) = \mathrm{T}3. \tag{173}$$

Consider now again the vertices of some fixed degree with exception of the polar ones; for any such vertex $V(\mathrm{T})$ the operation $\alpha^-(\mathrm{T},1)$ is defined, while the operations $\alpha^-(\mathrm{T},2)$, $\alpha^-(\mathrm{T},3)$ are defined with exception of the vertices $V(a0\mathrm{T}_{\{0,3\}})$, $V(a1\mathrm{T}_{\{0,2\}})$, respectively (in other words, they are undefined if $X_2(\sigma(0,\mathrm{T})) = 1$, $X_2(\sigma(1,\mathrm{T})) = 1$, respectively). For any $d \in \{1,2,3\}$, if the operation $\alpha^-(\mathrm{T},d)$ is defined, its result is unique. Using the formulae (150) – (159) and (162) – (171) we obtain after some calculation that for any $d \in \{1,2,3\}$

$$\alpha^+(\alpha^-(\mathrm{T},d),d) = \mathrm{T} \tag{174}$$

(if the inner operation is defined) and

$$\alpha^-(\alpha^+(\mathrm{T},d),d) = \mathrm{T} \tag{175}$$

(if the outer operation is defined and if the following two cases do not happen: $d = 2$ and T has the form $\alpha^-(a1\mathrm{T}_{\{0,2\}},1)1$ or $d = 3$ and T has the form $\alpha^-(a0\mathrm{T}_{\{0,3\}},1)1$).

Concluding we can write down all neighbouring vertices of degree N of the given vertex $V(\mathrm{T})$ of degree N:
– if T has not the form $a0\mathrm{T}_{\{0,3\}}$ or $a1\mathrm{T}_{\{0,2\}}$ or $00\mathrm{T}_{\{0\}}$ or $01\mathrm{T}_{\{0\}}$, the neighbouring vertices are $V(\alpha^-(\mathrm{T},1))$, $V(\alpha^-(\mathrm{T},2))$, $V(\alpha^-(\mathrm{T},3))$, $V(\alpha^+(\mathrm{T},1))$, $V(\alpha^+(\mathrm{T},2))$, $V(\alpha^+(\mathrm{T},3))$;
– if T has the form $a0\mathrm{T}_{\{0,3\}}$, the neighbouring vertices are $V(\alpha^-(\mathrm{T},1))$, $V(\alpha^-(\mathrm{T},3))$, $V(\alpha^+(\mathrm{T},1))$, $V(\alpha^+(\mathrm{T},2))$, $V(\alpha^+(\mathrm{T},3))$, and, if $\mathrm{T}_{\{0,3\}}$ contains at least one digit 3, also $V(\alpha^-(\alpha^-(\mathrm{T},3),1))$;
– if T has the form $a1\mathrm{T}_{\{0,2\}}$, the neighbouring vertices are $V(\alpha^-(\mathrm{T},1))$, $V(\alpha^-(\mathrm{T},2))$, $V(\alpha^+(\mathrm{T},1))$, $V(\alpha^+(\mathrm{T},2))$, $V(\alpha^+(\mathrm{T},3))$, and, if $\mathrm{T}_{\{0,2\}}$ contains at least one digit 2, also $V(\alpha^-(\alpha^-(\mathrm{T},2),1))$;
– if T has the form $00\mathrm{T}_{\{0\}}$, the neighbouring vertices are $V(a0\mathrm{T}_{\{3\}})$ for all $a \in \{1,2,3,4,5\}$, where $\mathrm{T}_{\{3\}}$ contains N digits 3;
– if T has the form $01\mathrm{T}_{\{0\}}$, the neighbouring vertices are $V(a1\mathrm{T}_{\{2\}})$ for all $a \in \{1,2,3,4,5\}$, where $\mathrm{T}_{\{2\}}$ contains N digits 2.

8. Discussion

The presented method of construction of the maximally regular net of domains can be used for any smooth surface, which does not differ too much from the spherical surface. Among such surfaces the most important case is the surface of the rotational ellipsoid, as many planetary bodies have approximately this shape. In the case of the spherical surface, the normal vector to the surface at each point of this surface is proportional to the radius-vector of this point of the surface (and thus to the unit vector $\boldsymbol{v}(P)$ of a point P lying at the unit sphere, see Section 2). However, in the case of the rotational ellipsoid, these two vectors have in general not the same direction, and we have several alternatives for the definition of the vector corresponding to $\boldsymbol{v}(P)$. It can be shown that the most natural definition is that using the ellipsoidal coordinates (coordinates of oblate spheroid) (see e.g. *Bateman* and *Erdélyi, 1953*, 16.1.3): the vector corresponding to $\boldsymbol{v}(P)$ will be the unit vector of the external normal to the ellipsoidal surface at infinity.

Comparing the presented method of denotation of domains and their vertices with the usual spherical coordinates, we see two peculiarities of our method: the first one is that our denotation uses a single sequence of digits compared to two angular coordinates (which are real numbers), the second one is that in our approach the domains are primary and the vertices secondary, on the contrary to the primary role of the points at the spherical surface in the usual approach. As in the practice the point is in fact not a mathematical point and its coordinates are given with certain precision, it is natural to represent such a point rather by a domain of sufficiently small size (thus of sufficiently large degree). This is exactly the aim of the presented method.

The formulae in this article, especially those in Sections 5 and 7, may seem to be complicated and cumbersome; however, by the machine evaluation they are extremely simple. In the case of the net of domains of large degree it is advantageous to calculate first the positions of all vertices of the given degree (these can be the 3 Cartesian coordinates) and store them in some suitable order for later use. This order may be the order defined naturally by the denotation of vertices or some other one based on the former: this is the reason for the elaborate definition of the unique coding of vertices in the Section 5.

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