A solution of the inverse problem of gravimetry for an ellipsoidal planetary body

V. Pohánka
Geophysical Institute of the Slovak Academy of Sciences[1]

Abstract: The inverse problem of gravimetry for a planetary body of the shape of a rotational ellipsoid is solved using the method described in *Pohánka (1997)*.

Key words: rotational ellipsoid, spherical functions

1. Introduction

In this paper we present the solution of the inverse problem of gravimetry for a planetary body of the shape of a rotational ellipsoid; this solution was obtained by applying the general method for a body of arbitrary shape published in *Pohánka (1997)*. The formulation of the inverse problem is similar to that presented in the mentioned paper: gravity field is assumed to be generated by the matter in the interior of a planetary body of the shape of a rotational ellipsoid; at the surface of the body the value of the normal derivative (with respect to the surface) of the gravity potential (and, in the case of the extended problem, also the surface value of density of the matter) is given (as input); the problem is to find every density function (from some given class of functions) generating the given external gravity field.

Let the equatorial and polar radius of the body be a and b, respectively, where

$$b = a\sqrt{1-\varepsilon^2} \tag{1.1}$$

[1]Dúbravská cesta 9, 842 28 Bratislava, Slovak Republic
e-mail: geofpohv@savba.sk

and $0 \le \varepsilon < 1$. We introduce the rectangular coordinate system with the origin in the centre of the body and the base vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ so that the unit vector $\boldsymbol{k}$ is parallel to the rotational axis of the body. In the corresponding spherical coordinate system with coordinates r, ϑ, φ we can write the radius vector $\boldsymbol{r}$ of an arbitrary point in space as

$$\boldsymbol{r} = r\ (\boldsymbol{i} \sin\vartheta \cos\varphi + \boldsymbol{j} \sin\vartheta \sin\varphi + \boldsymbol{k} \cos\vartheta). \tag{1.2}$$

Further we introduce the ellipsoidal coordinates (coordinates of oblate spheroid) υ, ξ, ψ by

$$\begin{aligned} r &= a\sqrt{(1-\varepsilon^2)\upsilon^2 + \varepsilon^2(\sin\xi)^2}, \\ \vartheta &= \arccos \frac{\sqrt{1-\varepsilon^2}\,\upsilon\cos\xi}{\sqrt{(1-\varepsilon^2)\upsilon^2 + \varepsilon^2(\sin\xi)^2}}, \\ \varphi &= \psi, \end{aligned} \tag{1.3}$$

where $\upsilon \ge 0$, $0 \le \xi \le \pi$, $0 \le \psi < 2\pi$ (see *Bateman* and *Erdélyi (1953)*, 16.1.3, and *Pohánka (1995)*, Section 6). These coordinates are defined in such a way that the surface of the body S is given by the condition $\upsilon = 1$ and the interior (exterior) domain D_{int} (D_{ext}) determined by the surface S is given by the condition $0 \le \upsilon < 1$ ($\upsilon > 1$).

From (1.2) and (1.3) we get for the radius vector $\boldsymbol{r}$ the expression

$$\begin{aligned} \boldsymbol{r} = a\Big(&\sqrt{(1-\varepsilon^2)\upsilon^2 + \varepsilon^2}\sin\xi\ (\boldsymbol{i}\cos\psi + \boldsymbol{j}\sin\psi) + \\ &+ \boldsymbol{k}\sqrt{1-\varepsilon^2}\,\upsilon\cos\xi\Big) \end{aligned} \tag{1.4}$$

and thus surface S is given in the parametrical form by

$$\boldsymbol{r} = \boldsymbol{s}(\xi, \psi), \tag{1.5}$$

where

$$\boldsymbol{s}(\xi, \psi) = a\ (\boldsymbol{i} \sin\xi \cos\psi + \boldsymbol{j} \sin\xi \sin\psi + \boldsymbol{k}\sqrt{1-\varepsilon^2}\cos\xi). \tag{1.6}$$

We accept the convention that the vector $\boldsymbol{r}$ satisfying (1.5) will be written as $\boldsymbol{s}$ (thus $\boldsymbol{s}$ is the radius vector of an arbitrary point of surface S). We further introduce the unit vector

$$\boldsymbol{v} = \boldsymbol{i}\sin\xi\cos\psi + \boldsymbol{j}\sin\xi\sin\psi + \boldsymbol{k}\cos\xi \tag{1.7}$$

and we can write any function $f(\xi,\psi)$ briefly as $f(\boldsymbol{v})$.

The unit vector $\boldsymbol{n}(\boldsymbol{s})$ of the external normal to the surface S at the point $\boldsymbol{s}$ is given by (see *Pohánka (1995)*, Section 2)

$$\boldsymbol{n}(\boldsymbol{s}) = \boldsymbol{n}(\boldsymbol{s}(\boldsymbol{v})) = \frac{\boldsymbol{o}(\boldsymbol{v})}{k(\boldsymbol{v})}, \tag{1.8}$$

where

$$\boldsymbol{o}(\boldsymbol{v}) = \boldsymbol{i}\sqrt{1-\varepsilon^2}\sin\xi\cos\psi + \boldsymbol{j}\sqrt{1-\varepsilon^2}\sin\xi\sin\psi + \boldsymbol{k}\cos\xi \tag{1.9}$$

and

$$k(\boldsymbol{v}) = |\boldsymbol{o}(\boldsymbol{v})| = \sqrt{1-\varepsilon^2(\sin\xi)^2}. \tag{1.10}$$

Following the conventions adopted in *Pohánka (1995)* and *Pohánka (1997)*, we denote the inner (outer) limit of the function $f(\boldsymbol{r})$ at the point $\boldsymbol{s}$ of surface S as $[f(\boldsymbol{s})]_{\text{int}}$ ($[f(\boldsymbol{s})]_{\text{ext}}$), and the normal component of the inner (outer) limit of the gradient of this function at the point $\boldsymbol{s}$ as $[\nu_{\boldsymbol{s}} f(\boldsymbol{s})]_{\text{int}} = \boldsymbol{n}(\boldsymbol{s}) \cdot [\nabla_{\boldsymbol{s}} f(\boldsymbol{s})]_{\text{int}}$ ($[\nu_{\boldsymbol{s}} f(\boldsymbol{s})]_{\text{ext}} = \boldsymbol{n}(\boldsymbol{s}) \cdot [\nabla_{\boldsymbol{s}} f(\boldsymbol{s})]_{\text{ext}}$).

In order to express this normal component of gradient in ellipsoidal coordinates, we first note that $\nabla_{\boldsymbol{r}}\boldsymbol{r}$ is the identity tensor and we have

$$\nabla_{\boldsymbol{r}}\boldsymbol{r} = \boldsymbol{ii} + \boldsymbol{jj} + \boldsymbol{kk} = (\nabla_{\boldsymbol{r}}\upsilon)\partial_\upsilon\boldsymbol{r} + (\nabla_{\boldsymbol{r}}\xi)\partial_\xi\boldsymbol{r} + (\nabla_{\boldsymbol{r}}\psi)\partial_\psi\boldsymbol{r}. \tag{1.11}$$

As it can be easily derived from (1.4), vectors $\partial_\upsilon\boldsymbol{r}, \partial_\xi\boldsymbol{r}, \partial_\psi\boldsymbol{r}$ are mutually orthogonal (this expresses the orthogonality of the ellipsoidal coordinate system). Then it follows from (1.11) that

$$\partial_\upsilon\boldsymbol{r} = (\nabla_{\boldsymbol{r}}\upsilon)\,(\partial_\upsilon\boldsymbol{r})^2, \quad \partial_\xi\boldsymbol{r} = (\nabla_{\boldsymbol{r}}\xi)\,(\partial_\xi\boldsymbol{r})^2, \quad \partial_\psi\boldsymbol{r} = (\nabla_{\boldsymbol{r}}\psi)\,(\partial_\psi\boldsymbol{r})^2, \tag{1.12}$$

and this implies that vectors $\nabla_{\boldsymbol{r}}\upsilon, \nabla_{\boldsymbol{r}}\xi, \nabla_{\boldsymbol{r}}\psi$ are also mutually orthogonal. Vector $\boldsymbol{n}(\boldsymbol{s}(\boldsymbol{v}))$ is proportional to the vector $\partial_\xi\boldsymbol{s}(\boldsymbol{v}) \times \partial_\psi\boldsymbol{s}(\boldsymbol{v})$ (see *Pohánka (1995)*, (2.6)) and thus it is orthogonal to vectors $[\nabla_{\boldsymbol{r}}\xi]_{\upsilon=1}$ and $[\nabla_{\boldsymbol{r}}\psi]_{\upsilon=1}$; therefore, it holds true that

$$\boldsymbol{n}(\boldsymbol{s}) \cdot [\nabla_{\boldsymbol{s}} f(\boldsymbol{s})]_{\text{int}} = \boldsymbol{n}(\boldsymbol{s}) \cdot [\nabla_{\boldsymbol{r}}\upsilon]_{\upsilon=1}\ \lim_{\upsilon\to1-}\ \partial_\upsilon f(\boldsymbol{r}). \tag{1.13}$$

From (1.12), (1.4), (1.9), and (1.10) we easily get

$$[\nabla_{\boldsymbol{r}} \upsilon]_{\upsilon=1} = \frac{1}{a\sqrt{1-\varepsilon^2}} \frac{\boldsymbol{o}(\boldsymbol{v})}{k(\boldsymbol{v})^2} \tag{1.14}$$

and using (1.8) we have

$$\boldsymbol{n}(\boldsymbol{s}) \cdot [\nabla_{\boldsymbol{r}} \upsilon]_{\upsilon=1} = \frac{1}{a\sqrt{1-\varepsilon^2}\, k(\boldsymbol{v})}. \tag{1.15}$$

Then we obtain from (1.13) and (1.15)

$$[\nu_{\boldsymbol{s}} f(\boldsymbol{s})]_{\text{int}} = \frac{1}{a\sqrt{1-\varepsilon^2}\, k(\boldsymbol{v})} \ \lim\nolimits_{\upsilon\to 1-} \partial_\upsilon f(\boldsymbol{r}); \tag{1.16}$$

similarly

$$[\nu_{\boldsymbol{s}} f(\boldsymbol{s})]_{\text{ext}} = \frac{1}{a\sqrt{1-\varepsilon^2}\, k(\boldsymbol{v})} \ \lim\nolimits_{\upsilon\to 1+} \partial_\upsilon f(\boldsymbol{r}). \tag{1.17}$$

2. Solution of the inverse problem

We follow here the method for finding the solution described in *Pohánka (1997)*: the potential of the gravity field $V(\boldsymbol{r})$ and the density of the matter $\rho(\boldsymbol{r})$ satisfy, outside the body, the equations

$$\boldsymbol{r} \in D_{\text{ext}}: \quad \rho(\boldsymbol{r}) = 0, \tag{2.1}$$

$$\boldsymbol{r} \in D_{\text{ext}}: \quad \Delta V(\boldsymbol{r}) = 0, \tag{2.2}$$

and the potential tends to zero at infinity; within the body they satisfy the Poisson equation

$$\boldsymbol{r} \in D_{\text{int}}: \quad \Delta V(\boldsymbol{r}) = 4\pi\kappa\, \rho(\boldsymbol{r}), \tag{2.3}$$

where κ is the gravitational constant. The potential and its gradient are continuous in the whole space; this implies that at the surface S we have the following continuity conditions:

$$[V(\boldsymbol{s})]_{\text{int}} = [V(\boldsymbol{s})]_{\text{ext}}, \tag{2.4}$$

$$[\nu_{\boldsymbol{s}} V(\boldsymbol{s})]_{\text{int}} = [\nu_{\boldsymbol{s}} V(\boldsymbol{s})]_{\text{ext}}. \tag{2.5}$$

The function on the r.h.s. of the last equation is the input of the inverse problem; this function will be denoted as $g(\boldsymbol{s})$, thus

$$[\nu_{\boldsymbol{s}} V(\boldsymbol{s})]_{\text{ext}} = g(\boldsymbol{s}). \tag{2.6}$$

Now we express potential $V(\boldsymbol{r})$ in the interior of the body in the form of

$$\boldsymbol{r} \in D_{\text{int}}: \quad V(\boldsymbol{r}) = U_0(\boldsymbol{r}) + Q(\boldsymbol{r})\, U_1(\boldsymbol{r}) + Q(\boldsymbol{r})^2\, W(\boldsymbol{r}), \tag{2.7}$$

where functions $Q(\boldsymbol{r})$, $U_0(\boldsymbol{r})$, $U_1(\boldsymbol{r})$ and $W(\boldsymbol{r})$ have, in domain D_{int}, bounded derivatives of the second order, and function $Q(\boldsymbol{r})$ satisfies the following conditions:

$$\boldsymbol{r} \in D_{\text{int}}: \quad Q(\boldsymbol{r}) > 0, \tag{2.8}$$

$$[Q(\boldsymbol{s})]_{\text{int}} = 0, \tag{2.9}$$

$$-\,[\nu_{\boldsymbol{s}} Q(\boldsymbol{s})]_{\text{int}} \geq c > 0, \tag{2.10}$$

where c is a suitable constant. Then it is clear that

$$\boldsymbol{n}(\boldsymbol{s}) = -\,\frac{1}{K(\boldsymbol{s})}\,[\nabla_{\boldsymbol{s}} Q(\boldsymbol{s})]_{\text{int}}, \tag{2.11}$$

where

$$K(\boldsymbol{s}) = |[\nabla_{\boldsymbol{s}} Q(\boldsymbol{s})]_{\text{int}}| \tag{2.12}$$

and thus

$$[\nu_{\boldsymbol{s}} Q(\boldsymbol{s})]_{\text{int}} = -\,K(\boldsymbol{s}). \tag{2.13}$$

Inserting expression (2.7) in conditions (2.4), (2.5), and using (2.9), (2.13) we get

$$[U_0(\boldsymbol{s})]_{\text{int}} = [V(\boldsymbol{s})]_{\text{ext}}, \tag{2.14}$$

$$[\nu_{\boldsymbol{s}} U_0(\boldsymbol{s})]_{\text{int}} - K(\boldsymbol{s})\,[U_1(\boldsymbol{s})]_{\text{int}} = [\nu_{\boldsymbol{s}} V(\boldsymbol{s})]_{\text{ext}}, \tag{2.15}$$

and functions $U_0(\boldsymbol{r})$ and $U_1(\boldsymbol{r})$ can be chosen to be harmonic:

$$\boldsymbol{r} \in D_{\text{int}}: \quad \Delta U_0(\boldsymbol{r}) = 0, \tag{2.16}$$

$$\boldsymbol{r} \in D_{\text{int}}: \quad \Delta U_1(\boldsymbol{r}) = 0. \tag{2.17}$$

Function $W(\boldsymbol{r})$ can be chosen arbitrarily (it only has to have, in domain D_{int}, bounded derivatives of the second order). According to (2.3) we then obtain the following expression for the density

$$\boldsymbol{r} \in D_{\text{int}}: \quad \rho(\boldsymbol{r}) = \frac{1}{4\pi\kappa}\, \Delta\Big(Q(\boldsymbol{r})\, U_1(\boldsymbol{r}) + Q(\boldsymbol{r})^2\, W(\boldsymbol{r})\Big) \tag{2.18}$$

and thus only the function $U_1(\boldsymbol{r})$ has to be calculated.

In the case when the surface value of the density (thus function $[\rho(\boldsymbol{s})]_{\text{int}}$) is also known, following *Pohánka (1997)* (Section 3), we rewrite function $W(\boldsymbol{r})$ in the form of

$$\boldsymbol{r} \in D_{\text{int}}: \quad W(\boldsymbol{r}) = U_2(\boldsymbol{r}) + Q(\boldsymbol{r})\, Z(\boldsymbol{r}), \tag{2.19}$$

where functions $U_2(\boldsymbol{r})$ and $Z(\boldsymbol{r})$ have, in domain D_{int}, bounded derivatives of the second order, and function $U_2(\boldsymbol{r})$ can be chosen to be harmonic:

$$\boldsymbol{r} \in D_{\text{int}}: \quad \Delta U_2(\boldsymbol{r}) = 0. \tag{2.20}$$

Then we get from (2.7)

$$\boldsymbol{r} \in D_{\text{int}}: \quad V(\boldsymbol{r}) = U_0(\boldsymbol{r}) + Q(\boldsymbol{r})\, U_1(\boldsymbol{r}) + Q(\boldsymbol{r})^2\, U_2(\boldsymbol{r}) + Q(\boldsymbol{r})^3\, Z(\boldsymbol{r}) \tag{2.21}$$

and instead of (2.18) we have the expression for the density

$$\boldsymbol{r} \in D_{\text{int}}: \quad \rho(\boldsymbol{r}) = \frac{1}{4\pi\kappa}\, \Delta\Big(Q(\boldsymbol{r})\, U_1(\boldsymbol{r}) + Q(\boldsymbol{r})^2\, U_2(\boldsymbol{r}) + Q(\boldsymbol{r})^3\, Z(\boldsymbol{r})\Big). \tag{2.22}$$

Taking the limit to surface S, we obtain after some calculation using (2.16), (2.17), (2.20), (2.9), and (2.11)

$$\begin{aligned} 4\pi\kappa\, [\rho(\boldsymbol{s})]_{\text{int}} &= [\Delta Q(\boldsymbol{s})]_{\text{int}}\, [U_1(\boldsymbol{s})]_{\text{int}} + 2\, [\nabla_{\boldsymbol{s}} Q(\boldsymbol{s})]_{\text{int}} \cdot [\nabla_{\boldsymbol{s}} U_1(\boldsymbol{s})]_{\text{int}} + \\ &\quad + 2\, [\nabla_{\boldsymbol{s}} Q(\boldsymbol{s})]_{\text{int}}^2\, [U_2(\boldsymbol{s})]_{\text{int}} = \\ &= L(\boldsymbol{s})\, [U_1(\boldsymbol{s})]_{\text{int}} - 2\, K(\boldsymbol{s})\, [\nu_{\boldsymbol{s}} U_1(\boldsymbol{s})]_{\text{int}} + \\ &\quad + 2\, K(\boldsymbol{s})^2\, [U_2(\boldsymbol{s})]_{\text{int}}, \end{aligned} \tag{2.23}$$

where we denoted

$$L(\boldsymbol{s}) = [\Delta Q(\boldsymbol{s})]_{\text{int}}. \tag{2.24}$$

As the condition (2.23) does not contain function $Z(\boldsymbol{r})$, this function can be chosen arbitrarily (it has only to have, in domain D_{int}, bounded derivatives of the second order); therefore, in this case, we have to calculate functions $U_1(\boldsymbol{r})$ and $U_2(\boldsymbol{r})$.

These functions are determined by the conditions (2.15) and (2.23); in view of (1.16) and (1.17) it is clear that the calculation of these functions will be simpler if we put

$$K(\boldsymbol{s}) = \frac{K}{a\sqrt{1-\varepsilon^2}\,k(\boldsymbol{v})}, \tag{2.25}$$

$$L(\boldsymbol{s}) = L\,K(\boldsymbol{s})^2, \tag{2.26}$$

where K, L are suitable constants. Formulae (2.25), (2.26) represent additional conditions imposed on function $Q(\boldsymbol{r})$; now we can determine this function.

It is clear that $Q(\boldsymbol{r})$ can be chosen to depend on $\boldsymbol{r}^2$ and $(\boldsymbol{k}\cdot\boldsymbol{r})^2$ only. Obviously, conditions (2.8) and (2.9) are satisfied if $Q(\boldsymbol{r})$ is equal to $E(\boldsymbol{r})$, where

$$E(\boldsymbol{r}) = 1 - \frac{\boldsymbol{r}^2}{a^2} - \frac{\varepsilon^2}{1-\varepsilon^2}\,\frac{(\boldsymbol{k}\cdot\boldsymbol{r})^2}{a^2}, \tag{2.27}$$

as in the ellipsoidal coordinates we have according to (1.4)

$$E(\boldsymbol{r}) = (1-\upsilon^2)\Big(1-\varepsilon^2(\sin\xi)^2\Big) \tag{2.28}$$

and thus

$$E(\boldsymbol{s}) = 0. \tag{2.29}$$

However, condition (2.25) is not satisfied if $Q(\boldsymbol{r})$ is equal to $E(\boldsymbol{r})$; therefore, we express function $Q(\boldsymbol{r})$ in the form of

$$Q(\boldsymbol{r}) = Q(E(\boldsymbol{r}), F(\boldsymbol{r})), \tag{2.30}$$

where function $F(\boldsymbol{r})$ is independent of $E(\boldsymbol{r})$. We choose

$$F(\boldsymbol{r}) = 1 - \varepsilon^2 + \frac{\varepsilon^2}{1-\varepsilon^2}\,\frac{(\boldsymbol{k}\cdot\boldsymbol{r})^2}{a^2}, \tag{2.31}$$

as in the ellipsoidal coordinates we have

$$F(\boldsymbol{r}) = 1 - \varepsilon^2 + \varepsilon^2 \upsilon^2 (\cos\xi)^2 \tag{2.32}$$

and thus

$$F(\boldsymbol{s}) = k(\boldsymbol{v})^2. \tag{2.33}$$

For maximal simplicity we require that $Q(e,f)$ is a rational function of variables e, f. We denote

$$\begin{aligned}
&Q_1(e,f) = \partial_e Q(e,f), \quad Q_2(e,f) = \partial_f Q(e,f),\\
&Q_{11}(e,f) = \partial_e^2 Q(e,f), \quad Q_{22}(e,f) = \partial_f^2 Q(e,f),\\
&Q_{12}(e,f) = \partial_e \partial_f Q(e,f),
\end{aligned} \tag{2.34}$$

and we get

$$\nabla_{\boldsymbol{r}} Q(\boldsymbol{r}) = Q_1(E(\boldsymbol{r}),F(\boldsymbol{r}))\ \nabla_{\boldsymbol{r}} E(\boldsymbol{r}) + Q_2(E(\boldsymbol{r}),F(\boldsymbol{r}))\ \nabla_{\boldsymbol{r}} F(\boldsymbol{r}), \tag{2.35}$$

$$\begin{aligned}
\Delta Q(\boldsymbol{r}) = {} & Q_1(E(\boldsymbol{r}),F(\boldsymbol{r}))\ \Delta E(\boldsymbol{r}) + Q_2(E(\boldsymbol{r}),F(\boldsymbol{r}))\ \Delta F(\boldsymbol{r}) + \\
& + Q_{11}(E(\boldsymbol{r}),F(\boldsymbol{r}))\ (\nabla_{\boldsymbol{r}} E(\boldsymbol{r}))^2 + Q_{22}(E(\boldsymbol{r}),F(\boldsymbol{r}))\ (\nabla_{\boldsymbol{r}} F(\boldsymbol{r}))^2 + \\
& + 2\,Q_{12}(E(\boldsymbol{r}),F(\boldsymbol{r}))\ (\nabla_{\boldsymbol{r}} E(\boldsymbol{r}))\cdot(\nabla_{\boldsymbol{r}} F(\boldsymbol{r})).
\end{aligned} \tag{2.36}$$

We have

$$\nabla_{\boldsymbol{r}} E(\boldsymbol{r}) = -\frac{2}{a^2}\left(\boldsymbol{r} + \frac{\varepsilon^2}{1-\varepsilon^2}\,(\boldsymbol{k}\cdot\boldsymbol{r})\,\boldsymbol{k}\right), \tag{2.37}$$

$$\nabla_{\boldsymbol{r}} F(\boldsymbol{r}) = \frac{2}{a^2}\,\frac{\varepsilon^2}{1-\varepsilon^2}\,(\boldsymbol{k}\cdot\boldsymbol{r})\,\boldsymbol{k} \tag{2.38}$$

and

$$\Delta E(\boldsymbol{r}) = -\frac{2}{a^2}\,\frac{3-2\varepsilon^2}{1-\varepsilon^2}, \tag{2.39}$$

$$\Delta F(\boldsymbol{r}) = \frac{2}{a^2}\,\frac{\varepsilon^2}{1-\varepsilon^2}; \tag{2.40}$$

further we easily calculate

$$(\nabla_{\boldsymbol{r}} E(\boldsymbol{r}))^2 = \frac{4}{a^2}\left(\frac{F(\boldsymbol{r})}{1-\varepsilon^2} - E(\boldsymbol{r})\right), \tag{2.41}$$

$$(\nabla_{\boldsymbol{r}} F(\boldsymbol{r}))^2 = \frac{4\,\varepsilon^2}{a^2}\left(\frac{F(\boldsymbol{r})}{1-\varepsilon^2} - 1\right), \tag{2.42}$$

$$(\nabla_{\boldsymbol{r}} E(\boldsymbol{r}))\cdot(\nabla_{\boldsymbol{r}} F(\boldsymbol{r})) = -\,\frac{4}{a^2}\left(\frac{F(\boldsymbol{r})}{1-\varepsilon^2} - 1\right). \tag{2.43}$$

In view of (2.29) and (2.33), condition (2.9) implies that $Q(0, k(\boldsymbol{v})^2) = 0$; as $Q(e, f)$ is a rational function, we have

$$Q(0, f) = 0 \tag{2.44}$$

and also

$$Q_2(0, f) = 0, \qquad Q_{22}(0, f) = 0. \tag{2.45}$$

Then we get from (2.35)

$$[\nabla_{\boldsymbol{s}} Q(\boldsymbol{s})]_{\mathrm{int}} = Q_1(0, k(\boldsymbol{v})^2)\; [\nabla_{\boldsymbol{s}} E(\boldsymbol{s})]_{\mathrm{int}} \tag{2.46}$$

and from (2.12) and (2.25) we obtain that $K \geq 0$, and

$$\begin{aligned}\frac{K^2}{a^2(1-\varepsilon^2)\,k(\boldsymbol{v})^2} &= ([\nabla_{\boldsymbol{s}} Q(\boldsymbol{s})]_{\mathrm{int}})^2 = Q_1(0, k(\boldsymbol{v})^2)^2\; ([\nabla_{\boldsymbol{s}} E(\boldsymbol{s})]_{\mathrm{int}})^2 = \\ &= \frac{4\,k(\boldsymbol{v})^2}{a^2(1-\varepsilon^2)}\; Q_1(0, k(\boldsymbol{v})^2)^2. \end{aligned} \tag{2.47}$$

Using (1.6) and (1.9) we get from (2.37)

$$[\nabla_{\boldsymbol{s}} E(\boldsymbol{s})]_{\mathrm{int}} = -\,\frac{2\,\boldsymbol{o}(\boldsymbol{v})}{a\sqrt{1-\varepsilon^2}} \tag{2.48}$$

and according to (1.8) we have

$$[\nu_{\boldsymbol{s}} E(\boldsymbol{s})]_{\mathrm{int}} = -\,\frac{2\,k(\boldsymbol{v})}{a\sqrt{1-\varepsilon^2}}. \tag{2.49}$$

In view of condition (2.10) and formulae (2.46) and (2.49), we get that $Q_1(0, k(\boldsymbol{v})^2) > 0$; therefore, from (2.47) we obtain that $K > 0$, and

$$Q_1(0,f) = \frac{K}{2f}, \tag{2.50}$$

thus

$$Q_{12}(0,f) = -\frac{K}{2f^2}. \tag{2.51}$$

Similarly, from (2.36) we get

$$\begin{aligned}[\Delta Q(\boldsymbol{s})]_{\text{int}} &= Q_1(0,k(\boldsymbol{v})^2)\ [\Delta E(\boldsymbol{s})]_{\text{int}} + Q_{11}(0,k(\boldsymbol{v})^2)\ [\nabla_{\boldsymbol{s}} E(\boldsymbol{s})]_{\text{int}}^2 + \\ &\quad + 2\,Q_{12}(0,k(\boldsymbol{v})^2)\ [\nabla_{\boldsymbol{s}} E(\boldsymbol{s})]_{\text{int}} \cdot [\nabla_{\boldsymbol{s}} F(\boldsymbol{s})]_{\text{int}}\end{aligned} \tag{2.52}$$

and from (2.24) – (2.26) we obtain

$$\begin{aligned}L\,\frac{K^2}{a^2(1-\varepsilon^2)\,k(\boldsymbol{v})^2} &= [\Delta Q(\boldsymbol{s})]_{\text{int}} = \\ &= -\frac{(3-2\varepsilon^2)K}{a^2(1-\varepsilon^2)\,k(\boldsymbol{v})^2} + \frac{4\,k(\boldsymbol{v})^2}{a^2(1-\varepsilon^2)}\,Q_{11}(0,k(\boldsymbol{v})^2) + \\ &\quad + \frac{4K}{a^2\,k(\boldsymbol{v})^4}\left(\frac{k(\boldsymbol{v})^2}{1-\varepsilon^2} - 1\right).\end{aligned} \tag{2.53}$$

Then we have

$$Q_{11}(0,f) = \frac{K^2L-(1+2\varepsilon^2)K}{4\,f^2} + \frac{(1-\varepsilon^2)K}{f^3} \tag{2.54}$$

Now we rewrite function $Q(e,f)$ as

$$Q(e,f) = e\,\frac{c_0+c_1e+c_2f+c_3e^2+c_4ef+c_5f^2}{d_0+d_1e+d_2f+d_3e^2+d_4ef+d_5f^2}, \tag{2.55}$$

thus condition (2.44) is satisfied, and inserting in (2.50) and (2.54) we get

$$\frac{c_0+c_2f+c_5f^2}{d_0+d_2f+d_5f^2} = \frac{K}{2f}, \tag{2.56}$$

$$\begin{aligned}&\frac{2(c_1+c_4f)}{d_0+d_2f+d_5f^2} - \frac{2(c_0+c_2f+c_5f^2)(d_1+d_4f)}{(d_0+d_2f+d_5f^2)^2} = \\ &= \frac{K^2L-(1+2\varepsilon^2)K}{4\,f^2} + \frac{(1-\varepsilon^2)K}{f^3}.\end{aligned} \tag{2.57}$$

From the first equation we get

$$c_0 f + c_2 f^2 + c_5 f^3 = \frac{K}{2}\,(d_0 + d_2 f + d_5 f^2) \tag{2.58}$$

and thus

$$d_0 = 0, \qquad c_0 = \frac{K}{2}\,d_2, \qquad c_2 = \frac{K}{2}\,d_5, \qquad c_5 = 0. \tag{2.59}$$

From the second equation we get

$$\begin{aligned} &2(c_1 f^2 + c_4 f^3) - K\,(d_1 f + d_4 f^2) = \\ &= \Big((1-\varepsilon^2)K + \frac{1}{4}\,(K^2 L - (1+2\varepsilon^2)K)\,f\Big)(d_2 + d_5 f) \end{aligned} \tag{2.60}$$

and thus

$$\begin{aligned} &d_1 = -\,(1-\varepsilon^2)\,d_5, \qquad d_2 = 0, \\ &c_1 = \frac{K}{2}\,d_4 + \frac{1}{8}\,(K^2 L - (1+2\varepsilon^2)K)\,d_5, \qquad c_4 = 0. \end{aligned} \tag{2.61}$$

Further, from (2.56) it follows that d_5 cannot be zero; therefore we can put

$$d_5 = 1. \tag{2.62}$$

Now we have

$$Q(e,f) = e\,\frac{c_1 e + c_2 f + c_3 e^2}{-\,(1-\varepsilon^2)\,e + d_3 e^2 + d_4 e f + f^2}; \tag{2.63}$$

the remaining coefficients may be chosen according to the following criteria:

(a) function $Q(\boldsymbol{r})$ should be as close to the function $E(\boldsymbol{r})$ as possible; particularly, for $\varepsilon = 0$ these functions should be equal;

(b) although function $Q(\boldsymbol{r})$ is not used in domain D_{ext}, it would be advantageous to require that condition $Q(\boldsymbol{r}) = 0$ is satisfied only at the surface S;

(c) function $Q(\boldsymbol{r})$ should be as simple as possible.

In view of the first criterion we require that the function $Q(\boldsymbol{r})/E(\boldsymbol{r})$ differs from 1 only in the second order with respect to variables $\boldsymbol{r}^2$ and $(\boldsymbol{k}\cdot\boldsymbol{r})^2$; in this way we obtain the following conditions:

$$\begin{aligned} &c_1 + (1-\varepsilon^2)\,c_2 + c_3 = -\,\varepsilon^2(1-\varepsilon^2) + d_3 + (1-\varepsilon^2)\,d_4, \\ &c_1 + 2\,c_3 = -\,(1-\varepsilon^2) + 2\,d_3 + (1-\varepsilon^2)\,d_4, \\ &c_1 - c_2 + 2\,c_3 = -\,3\,(1-\varepsilon^2) + 2\,d_3 - \varepsilon^2 d_4. \end{aligned} \tag{2.64}$$

Further it is reasonable to require that

$$c_3 = d_3; \tag{2.65}$$

then we get

$$c_1 = -(1-\varepsilon^2)(2-\varepsilon^2), \qquad c_2 = 1-\varepsilon^2, \qquad d_4 = -(1-\varepsilon^2). \tag{2.66}$$

Finally, we choose

$$d_3 = 2-\varepsilon^2 \tag{2.67}$$

(as condition (b) requires that $c_3 > (1-\varepsilon^2/2)^2$) and we have

$$Q(e,f) = e\,\frac{(2-\varepsilon^2)\,e^2 - (1-\varepsilon^2)(2-\varepsilon^2)\,e + (1-\varepsilon^2)\,f}{(2-\varepsilon^2)\,e^2 - (1-\varepsilon^2)(f+1)\,e + f^2}. \tag{2.68}$$

We denote

$$N(e,f) = (2-\varepsilon^2)\,e^2 - (1-\varepsilon^2)(2-\varepsilon^2)\,e + (1-\varepsilon^2)\,f, \tag{2.69}$$

$$M(e,f) = (f-1+\varepsilon^2)(f-(1-\varepsilon^2)\,e), \tag{2.70}$$

so that

$$Q(e,f) = e\,\frac{N(e,f)}{N(e,f)+M(e,f)}. \tag{2.71}$$

From definitions (2.27) and (2.31) we easily get

$$\begin{aligned} N(E(\boldsymbol{r}),F(\boldsymbol{r})) \;&=\; (2-\varepsilon^2)\Big(\varepsilon^2 - \frac{\boldsymbol{r}^2}{a^2} - \frac{\varepsilon^2}{1-\varepsilon^2}\,\frac{(\boldsymbol{k}\cdot\boldsymbol{r})^2}{a^2}\Big)\cdot \\ &\qquad\qquad\cdot\Big(1 - \frac{\boldsymbol{r}^2}{a^2} - \frac{\varepsilon^2}{1-\varepsilon^2}\,\frac{(\boldsymbol{k}\cdot\boldsymbol{r})^2}{a^2}\Big) + \\ &\quad + (1-\varepsilon^2)^2 + \varepsilon^2\,\frac{(\boldsymbol{k}\cdot\boldsymbol{r})^2}{a^2} \geq \\ &\geq (1-\varepsilon^2)^2\Big(1 - \frac{2-\varepsilon^2}{4}\Big) = \frac{1}{4}\,(2+\varepsilon^2)(1-\varepsilon^2)^2, \end{aligned} \tag{2.72}$$

$$M(E(\boldsymbol{r}),F(\boldsymbol{r})) = \varepsilon^2\,\frac{(\boldsymbol{k}\cdot\boldsymbol{r})^2}{a^2}\Big(\frac{\boldsymbol{r}^2}{a^2} + \frac{\varepsilon^2(2-\varepsilon^2)}{(1-\varepsilon^2)^2}\,\frac{(\boldsymbol{k}\cdot\boldsymbol{r})^2}{a^2}\Big) \geq 0 \tag{2.73}$$

and

$$\boldsymbol{r}\in D_{\rm int}: \quad M(E(\boldsymbol{r}),F(\boldsymbol{r})) \leq \varepsilon^2. \tag{2.74}$$

This shows that function $Q(\boldsymbol{r})$ given by (2.30) and (2.68) satisfies criteria (a) and (b).

Finally, constants K and L can be obtained from (2.59), (2.61), (2.62), and (2.66):

$$K = 2\,(1-\varepsilon^2), \qquad L = -\,\frac{3-2\varepsilon^2}{2\,(1-\varepsilon^2)} \tag{2.75}$$

(thus it holds true tat $K > 0$). We see that function $Q(\boldsymbol{r})$ satisfies conditions (2.8) and (2.9); according to (2.13), (2.25), and (2.75) condition (2.10) is also satisfied.

At this point we depart from the course of solution described in *Pohánka (1997)*, as the Dirichlet and Neumann boundary problems for the Laplace equation for the domain of the shape of rotational ellipsoid can be solved exactly. This is the consequence of the fact that the Laplace equation can be solved in the ellipsoidal coordinates by separation of variables and harmonic functions can be expressed as a series of ellipsoidal harmonics (see for example *Heiskanen* and *Moritz (1967)*, Chapter 1, or *Hobson (1931)*, Paragraph 252).

We first introduce spherical functions $\mathrm{Y}_{n,m}(\boldsymbol{v})$ (for definition see *Pohánka (1995)*, (3.9), (3.10)); they are nonzero only for $n \geq |m|$, they are complex (it holds true that $\mathrm{Y}^*_{n,m}(\boldsymbol{v}) = \mathrm{Y}_{n,-m}(\boldsymbol{v})$, where asterisk denotes the complex conjugation) and they form an orthonormal system: for $n \geq |m|, n' \geq |m'|$ it holds true that

$$\frac{1}{4\pi}\int \mathrm{d}\Xi\; \mathrm{Y}^*_{n,m}(\boldsymbol{v})\; \mathrm{Y}_{n',m'}(\boldsymbol{v}) = \delta_{n,n'}\,\delta_{m,m'}, \tag{2.76}$$

where $\mathrm{d}\Xi = \sin\xi\;\mathrm{d}\xi\mathrm{d}\psi$.

Spherical functions form a complete system of functions on the (unit) sphere: any function $f(\boldsymbol{v})$ continuous on the sphere can be expressed as a convergent series

$$f(\boldsymbol{v}) = \sum\nolimits_{n\geq 0}\sum\nolimits_{|m|\leq n} f_{n,m}\,\mathrm{Y}_{n,m}(\boldsymbol{v}), \tag{2.77}$$

where coefficients $f_{n,m}$ are given by

$$f_{n,m} = \frac{1}{4\pi} \int \mathrm{d}\Xi\, f(\boldsymbol{v})\, \mathrm{Y}^*_{n,m}(\boldsymbol{v}); \tag{2.78}$$

it is clear that for a real function $f(\boldsymbol{v})$ it holds true that $f^*_{n,m} = f_{n,-m}$.

The adopted abbreviate notation of sums is the following: $\sum_{n\geq k}$ ($\sum_{n\leq k}$) is the summation over n (the first variable in the condition) from k to ∞ (from $-\infty$ to k) and $\sum_{k\leq n\leq l}$ is the summation over n (the middle variable in the condition) from k to l if $k \leq l$, and zero otherwise ($\sum_{|n|\leq k}$ is the abbreviation of $\sum_{-k\leq n\leq k}$).

It can be shown (see *Pohánka (1995)*, Section 3) that any function $f(\boldsymbol{v})$ given in the form of (2.26) satisfies the bound

$$|f(\boldsymbol{v})| \leq \sum\nolimits_{n\geq 0} \|f_{n,*}\|, \tag{2.79}$$

where the norm is given by

$$\|f_{n,*}\| = \sqrt{(2n+1) \sum\nolimits_{|m|\leq n} |f_{n,m}|^2}. \tag{2.80}$$

For any function $f(\boldsymbol{v})$ such that the series on the r.h.s. of (2.28) converges (such a function will be said to belong to the class S), the series on the r.h.s. of (2.26) converges absolutely and uniformly.

Now we can express function $V(\boldsymbol{r})$ in domain D_{ext} as a series

$$V(\boldsymbol{r}) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} V_{n,m} \frac{Q_n^{|m|}(\mathrm{i}\,\kappa(\varepsilon)\,\upsilon)}{Q_n^{|m|}(\mathrm{i}\,\kappa(\varepsilon))}\, \mathrm{Y}_{n,m}(\boldsymbol{v}) \tag{2.81}$$

and functions $U_k(\boldsymbol{r})$ ($k = 0, 1, 2$) in domain D_{int} as a series

$$U_k(\boldsymbol{r}) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} U_{k;n,m} \frac{P_n^{|m|}(\mathrm{i}\,\kappa(\varepsilon)\,\upsilon)}{P_n^{|m|}(\mathrm{i}\,\kappa(\varepsilon))}\, \mathrm{Y}_{n,m}(\boldsymbol{v}), \tag{2.82}$$

where

$$\kappa(\varepsilon) = \frac{\sqrt{1-\varepsilon^2}}{\varepsilon}, \tag{2.83}$$

and $P_n^m(z)$ and $Q_n^m(z)$ are the associated Legendre functions of the first and second kind, respectively, (see *Bateman* and *Erdélyi (1953)*, Chapter 3) defined for complex z. To proceed further we first derive certain useful formulae involving the associated Legendre functions of imaginary argument.

3. Some properties of Legendre functions of imaginary argument

Functions $P_n^{|m|}(z)$ and $Q_n^{|m|}(z)$ (where $|m| \leq n$ and z is complex) are defined as

$$P_n^{|m|}(z) = \frac{(n+|m|)!}{n!} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}t \, (z + \sqrt{z^2-1} \cos t)^n \cos mt \tag{3.1}$$

(see *Bateman* and *Erdélyi (1953)*, 3.7.14) and

$$Q_n^{|m|}(z) = (-1)^m \frac{n!}{(n-|m|)!} \int_0^{\infty} \mathrm{d}t \, \frac{\mathrm{ch}\, mt}{(z + \sqrt{z^2-1}\,\mathrm{ch}\, t)^{n+1}} \tag{3.2}$$

(see *Bateman* and *Erdélyi (1953)*, 3.7.12); they are regular in the whole z-plane with exception of the cut from -1 to 1. We are interested only in the functions $P_n^{|m|}(\mathrm{i}u)$, $Q_n^{|m|}(\mathrm{i}u)$ for $|m| \leq n$, u real, $u \geq 0$; in order to remove complex quantities we express them in the form of

$$P_n^{|m|}(\mathrm{i}u) = \mathrm{i}^n \, \mathrm{p}_n^{|m|}(u), \tag{3.3}$$

$$Q_n^{|m|}(\mathrm{i}u) = \frac{(-1)^m}{\mathrm{i}^{n+1}} \, \mathrm{q}_n^{|m|}(u), \tag{3.4}$$

where, as it follows from (3.1) and (3.2),

$$\mathrm{p}_n^{|m|}(u) = \frac{(n+|m|)!}{n!} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}t \, (u + \sqrt{u^2+1} \cos t)^n \cos mt, \tag{3.5}$$

$$\mathrm{q}_n^{|m|}(u) = \frac{n!}{(n-|m|)!} \int_0^{\infty} \mathrm{d}t \, \frac{\mathrm{ch}\, mt}{(u + \sqrt{u^2+1}\,\mathrm{ch}\, t)^{n+1}}. \tag{3.6}$$

In the sequel we shall use functions $\mathrm{p}_n^{|m|}(u)$ and $\mathrm{q}_n^{|m|}(u)$ only in the case when $|m| \leq n$, $u \geq 0$. Then it is clear from (3.6) that functions $\mathrm{q}_n^{|m|}(u)$ are always positive. In the case of functions $\mathrm{p}_n^{|m|}(u)$ we first write (3.5) in the form of

$$\mathrm{p}_n^{|m|}(u) = \frac{(n+|m|)!}{n!} \sum\nolimits_{0\le k\le n} \binom{n}{k} u^k \sqrt{u^2+1}^{\,n-k} \cdot \\ \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}t \, (\cos t)^{n-k} \cos mt; \qquad (3.7)$$

for $0 \le k \le n$ we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}t \, (\cos t)^{n-k} \cos mt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}t \, \frac{1}{2^{n-k}} \, (\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t})^{n-k} \, \mathrm{e}^{\mathrm{i}|m|t} = \\ = \frac{1}{2^{n-k}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}t \sum\nolimits_{0\le l\le n-k} \binom{n-k}{l} \mathrm{e}^{-\mathrm{i}(n-k-|m|-2l)t} = \\ = \frac{1}{2^{n-k}} \sum\nolimits_{0\le l\le n} \binom{n-k}{l} \delta_{n-|m|-2l,k} \qquad (3.8)$$

and thus

$$\mathrm{p}_n^{|m|}(u) = \frac{(n+|m|)!}{n!} \sum\nolimits_{0\le k\le n} \binom{n}{k} u^k \sqrt{u^2+1}^{\,n-k} \cdot \\ \cdot \frac{1}{2^{n-k}} \sum\nolimits_{0\le l\le n} \binom{n-k}{l} \delta_{n-|m|-2l,k} = \\ = \frac{(n+|m|)!}{n!} \sum\nolimits_{0\le l\le n} \sum\nolimits_{0\le k\le n} \delta_{n-|m|-2l,k} \binom{n}{|m|+2l} \binom{|m|+2l}{l} \cdot \\ \cdot \frac{1}{2^{|m|+2l}} u^{n-|m|-2l} \sqrt{u^2+1}^{\,|m|+2l} = \\ = \frac{(n+|m|)!}{n!} \sum\nolimits_{0\le l\le [(n-|m|)/2]} \binom{n}{|m|+2l} \binom{|m|+2l}{l} \cdot \\ \cdot \frac{1}{2^{|m|+2l}} u^{n-|m|-2l} \sqrt{u^2+1}^{\,|m|+2l}, \qquad (3.9)$$

where $[x]$ is the integer part of x. This expression shows that functions $\mathrm{p}_n^{|m|}(u)$ are positive with the exception of the case that $u = 0$ and $n - |m|$ is odd (when they are zero).

Now we are able to derive certain inequalities involving functions $\mathrm{p}_n^{|m|}(u)$

and $\mathrm{q}_n^{|m|}(u)$. We write (3.9) in the form of

$$\frac{\mathrm{p}_n^{|m|}(u)}{\sqrt{u^2+1}^{\,n}} = \frac{(n+|m|)!}{n!} \sum\nolimits_{0\le l\le[(n-|m|)/2]} \binom{n}{|m|+2l} \binom{|m|+2l}{l} \cdot \frac{1}{2^{|m|+2l}} \left(\frac{u}{\sqrt{u^2+1}}\right)^{n-|m|-2l} ; \quad (3.10)$$

as the quantity $u/\sqrt{u^2+1}$ is an increasing function of u, the expression on the r.h.s. is a nondecreasing function of u. Similarly, we write (3.6) in the form of

$$\mathrm{q}_n^{|m|}(u)\sqrt{u^2+1}^{\,n+1} = \frac{n!}{(n-|m|)!} \int_0^\infty \mathrm{d}t \, \frac{\mathrm{ch}\, mt}{(u/\sqrt{u^2+1}+\mathrm{ch}\, t)^{n+1}} \quad (3.11)$$

and we see that the expression on the r.h.s. is a nonincreasing function of u. Therefore it holds true that (see also *Hobson (1931)*, Paragraph 252)

$$0 \le u \le u',\ 0 < u' : \quad \frac{\mathrm{p}_n^{|m|}(u)}{\mathrm{p}_n^{|m|}(u')} \le \left(\frac{\sqrt{u^2+1}}{\sqrt{u'^2+1}}\right)^n , \quad (3.12)$$

$$0 \le u \le u' : \quad \frac{\mathrm{q}_n^{|m|}(u')}{\mathrm{q}_n^{|m|}(u)} \le \left(\frac{\sqrt{u^2+1}}{\sqrt{u'^2+1}}\right)^{n+1} . \quad (3.13)$$

Further, from (3.9) we get

$$\begin{aligned} u\,\partial_u \mathrm{p}_n^{|m|}(u) &= \frac{(n+|m|)!}{n!} \sum\nolimits_{0\le l\le[(n-|m|)/2]} \binom{n}{|m|+2l} \binom{|m|+2l}{l} \cdot \\ &\qquad \cdot \frac{1}{2^{|m|+2l}}\, u\,\partial_u u^{n-|m|-2l} \sqrt{u^2+1}^{\,|m|+2l} = \\ &= \frac{(n+|m|)!}{n!} \sum\nolimits_{0\le l\le[(n-|m|)/2]} \binom{n}{|m|+2l} \binom{|m|+2l}{l} \cdot \\ &\qquad \cdot \frac{1}{2^{|m|+2l}}\, u^{n-|m|-2l} \sqrt{u^2+1}^{\,|m|+2l} \cdot \\ &\qquad \cdot \left(n-|m|-2l + (|m|+2l)\, \frac{u^2}{u^2+1}\right) ; \quad (3.14) \end{aligned}$$

for $0 \le l \le [(n-|m|)/2]$ it holds true that

$$n\,\frac{u^2}{u^2+1} \le n-|m|-2l + (|m|+2l)\,\frac{u^2}{u^2+1} \le n \tag{3.15}$$

and comparing (3.14) with (3.9) we get the inequality

$$n\,\frac{u^2}{u^2+1}\,\mathrm{p}_n^{|m|}(u) \le u\,\partial_u \mathrm{p}_n^{|m|}(u) \le n\,\mathrm{p}_n^{|m|}(u). \tag{3.16}$$

Similarly, from (3.6) we get

$$\begin{aligned} -\,u\,\partial_u \mathrm{q}_n^{|m|}(u) &= \frac{n!}{(n-|m|)!}\int_0^\infty \mathrm{d}t\,(-\,u)\partial_u \frac{\operatorname{ch} mt}{(u+\sqrt{u^2+1}\operatorname{ch} t)^{n+1}} = \\ &= \frac{n!}{(n-|m|)!}\int_0^\infty \mathrm{d}t\,\frac{\operatorname{ch} mt}{(u+\sqrt{u^2+1}\operatorname{ch} t)^{n+2}}\,\cdot \\ &\qquad\qquad \cdot\,(n+1)\left(u+\frac{u^2}{\sqrt{u^2+1}}\operatorname{ch} t\right); \end{aligned} \tag{3.17}$$

it holds true that

$$\frac{u^2}{u^2+1}\,(u+\sqrt{u^2+1}\operatorname{ch} t) \le u + \frac{u^2}{\sqrt{u^2+1}}\operatorname{ch} t \le u+\sqrt{u^2+1}\operatorname{ch} t \tag{3.18}$$

and comparing (3.17) with (3.6) we get the inequality

$$(n+1)\,\frac{u^2}{u^2+1}\,\mathrm{q}_n^{|m|}(u) \le -\,u\,\partial_u \mathrm{q}_n^{|m|}(u) \le (n+1)\,\mathrm{q}_n^{|m|}(u). \tag{3.19}$$

After denoting

$$\partial \mathrm{p}_n^{|m|}(u) = \partial_u \mathrm{p}_n^{|m|}(u), \tag{3.20}$$

$$\partial \mathrm{q}_n^{|m|}(u) = \partial_u \mathrm{q}_n^{|m|}(u), \tag{3.21}$$

and using the above established positivity of functions $\mathrm{p}_n^{|m|}(u)$ and $\mathrm{q}_n^{|m|}(u)$, we obtain the inequalities

$$u>0: \qquad n\,\frac{u^2}{u^2+1} \le \frac{u\,\partial \mathrm{p}_n^{|m|}(u)}{\mathrm{p}_n^{|m|}(u)} \le n, \tag{3.22}$$

$$u\ge 0: \qquad (n+1)\,\frac{u^2}{u^2+1} \le -\,\frac{u\,\partial \mathrm{q}_n^{|m|}(u)}{\mathrm{q}_n^{|m|}(u)} \le n+1. \tag{3.23}$$

Finally, from (3.5) and (3.6) we have ($|m| \leq n$)

$$\frac{(n-|m|)!}{(n+|m|)!}\, \mathrm{p}_n^{|m|}(u)\, \mathrm{q}_n^{|m|}(u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\tau\, (u+\sqrt{u^2+1}\,\cos\tau)^n \cos m\tau \,\cdot \\ \cdot \int_0^{\infty} \mathrm{d}t\, \frac{\mathrm{ch}\, mt}{(u+\sqrt{u^2+1}\,\mathrm{ch}\, t)^{n+1}} = \\ = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\tau\, (u+\sqrt{u^2+1}\,\cos\tau)^n \mathrm{e}^{-\mathrm{i}m\tau} \,\cdot \\ \cdot \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t\, \frac{\mathrm{e}^{mt}}{(u+\sqrt{u^2+1}\,\mathrm{ch}\, t)^{n+1}} \tag{3.24}$$

and thus

$$\sum\nolimits_{|m|\leq n} \frac{(n-|m|)!}{(n+|m|)!}\, \mathrm{p}_n^{|m|}(u)\, \mathrm{q}_n^{|m|}(u) = \\ = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t\, \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\tau\, \frac{(u+\sqrt{u^2+1}\,\cos\tau)^n}{(u+\sqrt{u^2+1}\,\mathrm{ch}\, t)^{n+1}} \sum\nolimits_{|m|\leq n} \mathrm{e}^{-\mathrm{i}m\tau}\mathrm{e}^{mt}. \tag{3.25}$$

It is clear that for any function $f(\tau)$ having the form of

$$f(\tau) = \sum\nolimits_{|m|\leq n} f_m\, \mathrm{e}^{\mathrm{i}m\tau} \tag{3.26}$$

it holds true that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\tau\, f(\tau) \sum\nolimits_{|m|\leq n} \mathrm{e}^{-\mathrm{i}m\tau}\mathrm{e}^{mt} = \sum\nolimits_{|m|\leq n} f_m\, \mathrm{e}^{mt} = f(-\mathrm{i}t); \tag{3.27}$$

as $(u+\sqrt{u^2+1}\,\cos\tau)^n$ is such a function, we have

$$\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t\, \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\tau\, \frac{(u+\sqrt{u^2+1}\,\cos\tau)^n}{(u+\sqrt{u^2+1}\,\mathrm{ch}\, t)^{n+1}} \sum\nolimits_{|m|\leq n} \mathrm{e}^{-\mathrm{i}m\tau}\mathrm{e}^{mt} = \\ = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t\, \frac{(u+\sqrt{u^2+1}\,\mathrm{ch}\, t)^n}{(u+\sqrt{u^2+1}\,\mathrm{ch}\, t)^{n+1}} = \int_0^{\infty} \mathrm{d}t\, \frac{1}{u+\sqrt{u^2+1}\,\mathrm{ch}\, t} = \\ = \mathrm{q}_0^0(u) \tag{3.28}$$

and thus we obtain the formula

$$\sum\nolimits_{|m|\leq n} \frac{(n-|m|)!}{(n+|m|)!}\, \mathrm{p}_n^{|m|}(u)\, \mathrm{q}_n^{|m|}(u) = \mathrm{q}_0^0(u). \tag{3.29}$$

Therefore, using the above established positivity of functions $\mathrm{p}_n^{|m|}(u)$ and $\mathrm{q}_n^{|m|}(u)$, we get the inequality ($|m| \le n$)

$$0 \le \frac{(n-|m|)!}{(n+|m|)!}\, \mathrm{p}_n^{|m|}(u)\, \mathrm{q}_n^{|m|}(u) \le \mathrm{q}_0^0(u). \tag{3.30}$$

4. Derivation of the solution

Inserting (3.4) in (2.81) we get for $\boldsymbol{r} \in D_{\mathrm{ext}}$

$$V(\boldsymbol{r}) = \sum\nolimits_{n\ge 0} \sum\nolimits_{|m|\le n} V_{n,m}\, \frac{\mathrm{q}_n^{|m|}(\kappa(\varepsilon)\, \upsilon)}{\mathrm{q}_n^{|m|}(\kappa(\varepsilon))}\, \mathrm{Y}_{n,m}(\boldsymbol{v}) \tag{4.1}$$

and inserting (3.3) in (2.82) we get for $\boldsymbol{r} \in D_{\mathrm{int}}$

$$U_k(\boldsymbol{r}) = \sum\nolimits_{n\ge 0} \sum\nolimits_{|m|\le n} U_{k;n,m}\, \frac{\mathrm{p}_n^{|m|}(\kappa(\varepsilon)\, \upsilon)}{\mathrm{p}_n^{|m|}(\kappa(\varepsilon))}\, \mathrm{Y}_{n,m}(\boldsymbol{v}). \tag{4.2}$$

At the surface S we have (yet only formally)

$$[V(\boldsymbol{s})]_{\mathrm{ext}} = \sum\nolimits_{n\ge 0} \sum\nolimits_{|m|\le n} V_{n,m}\, \mathrm{Y}_{n,m}(\boldsymbol{v}), \tag{4.3}$$

$$[U_k(\boldsymbol{s})]_{\mathrm{int}} = \sum\nolimits_{n\ge 0} \sum\nolimits_{|m|\le n} U_{k;n,m}\, \mathrm{Y}_{n,m}(\boldsymbol{v}). \tag{4.4}$$

If the coefficients $V_{n,m}$ and $U_{k;n,m}$ are such that the series $\sum_{n\ge 0} \|V_{n,*}\|$ and $\sum_{n\ge 0} \|U_{k;n,*}\|$ converge, then (see Section 2) the series on the r.h.s. of (4.3) and (4.4) converge absolutely and uniformly. Inequalities (3.12) and (3.13) yield

$$0 \le \upsilon \le 1: \quad \frac{\mathrm{p}_n^{|m|}(\kappa(\varepsilon)\, \upsilon)}{\mathrm{p}_n^{|m|}(\kappa(\varepsilon))} \le \sqrt{(1-\varepsilon^2)\upsilon^2 + \varepsilon^2}^{\,n} \le 1, \tag{4.5}$$

$$\upsilon \ge 1: \quad \frac{\mathrm{q}_n^{|m|}(\kappa(\varepsilon)\, \upsilon)}{\mathrm{q}_n^{|m|}(\kappa(\varepsilon))} \le \frac{1}{\sqrt{(1-\varepsilon^2)\upsilon^2 + \varepsilon^2}^{\,n+1}} \le 1, \tag{4.6}$$

and this implies that also the series on the r.h.s. of (4.1) and (4.2) converge absolutely and uniformly (in this case also with respect to variable υ, where $\upsilon \geq 1$ and $0 \leq \upsilon \leq 1$, respectively); as a consequence, equalities (4.3) and (4.4) are true.

Further, if the coefficients $V_{n,m}$ and $U_{k;n,m}$ are such that also the series $\sum_{n\geq 0} \|(n+1)V_{n,*}\|$ and $\sum_{n\geq 0} \|nU_{k;n,*}\|$ converge, then the series, obtained by differentiation (term by term) of the series on the r.h.s. of (4.1) and (4.2) with respect to υ, converge absolutely and uniformly (for $\upsilon \geq 1$ and $0 \leq \upsilon \leq 1$, respectively): this follows from the inequalities (3.22) and (3.23). Therefore, according to (1.16) and (1.17) it holds true that

$$[\nu_{\boldsymbol{s}} V(\boldsymbol{s})]_{\text{ext}} = \\ = \frac{1}{a\sqrt{1-\varepsilon^2}\, k(\boldsymbol{v})} \sum_{n\geq 0} \sum_{|m|\leq n} V_{n,m} \frac{\kappa(\varepsilon)\, \partial \mathrm{q}_n^{|m|}(\kappa(\varepsilon))}{\mathrm{q}_n^{|m|}(\kappa(\varepsilon))} \,\mathrm{Y}_{n,m}(\boldsymbol{v}), \tag{4.7}$$

$$[\nu_{\boldsymbol{s}} U_k(\boldsymbol{s})]_{\text{int}} = \\ = \frac{1}{a\sqrt{1-\varepsilon^2}\, k(\boldsymbol{v})} \sum_{n\geq 0} \sum_{|m|\leq n} U_{k;n,m} \frac{\kappa(\varepsilon)\, \partial \mathrm{p}_n^{|m|}(\kappa(\varepsilon))}{\mathrm{p}_n^{|m|}(\kappa(\varepsilon))} \,\mathrm{Y}_{n,m}(\boldsymbol{v}). \tag{4.8}$$

Now we can determine coefficients $U_{0;n,m}$ and $U_{1;n,m}$ using conditions (2.14), (2.15), and formula (2.6). We first denote

$$G(\boldsymbol{s}) = a\sqrt{1-\varepsilon^2}\, k(\boldsymbol{v})\, g(\boldsymbol{s}) \tag{4.9}$$

and write function $G(\boldsymbol{s})$ in the form of a series

$$G(\boldsymbol{s}) = \sum_{n\geq 0} \sum_{|m|\leq n} G_{n,m}\, \mathrm{Y}_{n,m}(\boldsymbol{v}); \tag{4.10}$$

from (2.6) and (4.7) we then obtain (for $|m| \leq n$)

$$G_{n,m} = \frac{\kappa(\varepsilon)\, \partial \mathrm{q}_n^{|m|}(\kappa(\varepsilon))}{\mathrm{q}_n^{|m|}(\kappa(\varepsilon))}\, V_{n,m}. \tag{4.11}$$

Inequality (3.23) yields

$$(n+1)\,(1-\varepsilon^2) \leq -\frac{\kappa(\varepsilon)\, \partial \mathrm{q}_n^{|m|}(\kappa(\varepsilon))}{\mathrm{q}_n^{|m|}(\kappa(\varepsilon))} \leq n+1 \tag{4.12}$$

and thus we have

$$V_{n,m} = \frac{\mathrm{q}_n^{|m|}(\kappa(\varepsilon))}{\kappa(\varepsilon)\,\partial\mathrm{q}_n^{|m|}(\kappa(\varepsilon))}\, G_{n,m}. \tag{4.13}$$

Similarly, from (2.14), and (4.3), (4.4) we get (for $|m| \le n$)

$$U_{0;n,m} = V_{n,m}. \tag{4.14}$$

Therefore, if the function $G(\boldsymbol{s})$ belongs to the class S (thus the series $\sum_{n\ge0} \|G_{n,*}\|$ converges), the series $\sum_{n\ge0} \|(n{+}1)V_{n,*}\|$ and $\sum_{n\ge0} \|nU_{0;n,*}\|$ also converge; this means that equalities (4.3), (4.7), (4.4), (4.8) (the latter two for $k=0$), and also (4.13) and (4.14) are true.

Formula (2.15) together with (2.6) yields

$$K(\boldsymbol{s})\,[U_1(\boldsymbol{s})]_{\mathrm{int}} = [\nu_{\boldsymbol{s}} U_0(\boldsymbol{s})]_{\mathrm{int}} - g(\boldsymbol{s}); \tag{4.15}$$

using (4.4), (4.8), (4.9), (4.10), and (2.25) we get (for $|m| \le n$)

$$K\,U_{1;n,m} = \frac{\kappa(\varepsilon)\,\partial\mathrm{p}_n^{|m|}(\kappa(\varepsilon))}{\mathrm{p}_n^{|m|}(\kappa(\varepsilon))}\, U_{0;n,m} - G_{n,m} \tag{4.16}$$

and using (4.13), (4.14), and (2.75) we obtain

$$U_{1;n,m} = \frac{1}{2\,(1-\varepsilon^2)} \left(\frac{\mathrm{q}_n^{|m|}(\kappa(\varepsilon))}{\partial\mathrm{q}_n^{|m|}(\kappa(\varepsilon))}\, \frac{\partial\mathrm{p}_n^{|m|}(\kappa(\varepsilon))}{\mathrm{p}_n^{|m|}(\kappa(\varepsilon))} - 1 \right) G_{n,m}. \tag{4.17}$$

From (3.22) we have

$$n\,(1-\varepsilon^2) \le \frac{\kappa(\varepsilon)\,\partial\mathrm{p}_n^{|m|}(\kappa(\varepsilon))}{\mathrm{p}_n^{|m|}(\kappa(\varepsilon))} \le n \tag{4.18}$$

and using (4.12) we get the bound

$$\frac{n\,(1-\varepsilon^2)}{n{+}1} \le - \frac{\mathrm{q}_n^{|m|}(\kappa(\varepsilon))}{\partial\mathrm{q}_n^{|m|}(\kappa(\varepsilon))}\, \frac{\partial\mathrm{p}_n^{|m|}(\kappa(\varepsilon))}{\mathrm{p}_n^{|m|}(\kappa(\varepsilon))} \le \frac{n}{(n{+}1)\,(1-\varepsilon^2)}. \tag{4.19}$$

Thus, if the function $G(\boldsymbol{s})$ belongs to the class S, the series $\sum_{n\geq 0}\|U_{1;n,*}\|$ also converges; this means that equalities (4.4) (for $k=1$), (4.16), and (4.17) are true. Moreover, if also the series $\sum_{n\geq 0}\|nG_{n,*}\|$ converges, the same is true for the series $\sum_{n\geq 0}\|nU_{1;n,*}\|$; thus equality (4.8) holds true also for $k=1$.

In a similar way we can determine coefficients $U_{2;n,m}$ using condition (2.23). We have

$$2\,K(\boldsymbol{s})^2\,[U_2(\boldsymbol{s})]_{\rm int} = 4\pi\kappa\,[\rho(\boldsymbol{s})]_{\rm int} - L(\boldsymbol{s})\,[U_1(\boldsymbol{s})]_{\rm int} + +2\,K(\boldsymbol{s})\,[\nu_{\boldsymbol{s}}U_1(\boldsymbol{s})]_{\rm int}; \tag{4.20}$$

we denote

$$R(\boldsymbol{s}) = 4\pi\kappa\,a^2(1-\varepsilon^2)\,k(\boldsymbol{v})^2\,[\rho(\boldsymbol{s})]_{\rm int} \tag{4.21}$$

and write function $R(\boldsymbol{s})$ in the form of a series

$$R(\boldsymbol{s}) = \sum\nolimits_{n\geq 0}\sum\nolimits_{|m|\leq n} R_{n,m}\,\mathrm{Y}_{n,m}(\boldsymbol{v}). \tag{4.22}$$

Then we get using (4.4), (4.8), (2.25), and (2.26) (for $|m|\leq n$)

$$2\,K^2\,U_{2;n,m} = R_{n,m} - L\,K^2\,U_{1;n,m} + 2\,K\,\frac{\kappa(\varepsilon)\,\partial\mathrm{p}_n^{|m|}(\kappa(\varepsilon))}{\mathrm{p}_n^{|m|}(\kappa(\varepsilon))}\,U_{1;n,m} \tag{4.23}$$

and using (2.75) we obtain

$$U_{2;n,m} = \frac{1}{8\,(1-\varepsilon^2)^2}\,R_{n,m} + + \frac{1}{4\,(1-\varepsilon^2)}\left(2\,\frac{\kappa(\varepsilon)\,\partial\mathrm{p}_n^{|m|}(\kappa(\varepsilon))}{\mathrm{p}_n^{|m|}(\kappa(\varepsilon))} + 3 - 2\,\varepsilon^2\right)U_{1;n,m}. \tag{4.24}$$

In view of (4.18), if the series $\sum_{n\geq 0}\|nG_{n,*}\|$ converges and function $R(\boldsymbol{s})$ belongs to the class S (thus the series $\sum_{n\geq 0}\|R_{n,*}\|$ converges), the series $\sum_{n\geq 0}\|U_{2;n,*}\|$ also converges; this means that equalities (4.4) (for $k=2$) and (4.24) are true.

The solution of the inverse problem can be thus obtained as follows. For the given function $g(\boldsymbol{s})$, from (4.9) we get function $G(\boldsymbol{s})$ and calculate coefficients $G_{n,m}$ by inversion of equality (4.10) (see (2.77) and (2.78)). Then we get coefficients $U_{1;n,m}$ from (4.17) and function $U_1(\boldsymbol{r})$ is given by (4.2) (for $k=1$). Density $\rho(\boldsymbol{r})$ can be calculated from (2.18) for some choice of the function $W(\boldsymbol{r})$.

In the case that function $[\rho(\boldsymbol{s})]_{\text{int}}$ is also known, we first calculate coefficients $U_{1;n,m}$ and function $U_1(\boldsymbol{r})$ as before. Then we get function $R(\boldsymbol{s})$ from (4.21) and calculate coefficients $R_{n,m}$ by inversion of (4.22). Coefficients $U_{2;n,m}$ can be calculated from (4.24) and function $U_2(\boldsymbol{r})$ is given by (4.2) (for $k=2$). Density $\rho(\boldsymbol{r})$ can be calculated from (2.22) for some choice of the function $Z(\boldsymbol{r})$.

Finally, we note that the presented method for the solution of the inverse problem of gravimetry for an ellipsoidal planetary body is a generalization of the method treated in *Pohánka (1993)* for the body of spherical shape.

5. Improvement of the solution

In previous Sections the solution of the inverse problem of gravimetry was presented in the form that need not be the most suitable for numerical calculation. Therefore, we shall present certain modifications of this solution that can improve the speed of numerical calculation.

First, we derive a simpler expression of the fraction on the r.h.s. of (4.17). We use the formula (see *Hobson (1931)*, Par. 251)

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{\mathrm{i}}{a\,\varepsilon} \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} (-1)^m \, \frac{(n-|m|)!}{(n+|m|)!} \cdot \\ \cdot P_n^{|m|}(\mathrm{i}\,\kappa(\varepsilon)\,\upsilon)\; Q_n^{|m|}(\mathrm{i}\,\kappa(\varepsilon)\,\upsilon')\; \mathrm{Y}_{n,m}(\boldsymbol{v})\, \mathrm{Y}^*_{n,m}(\boldsymbol{v}'), \qquad (5.1)$$

where $0 \leq \upsilon < \upsilon'$ (note that in *Hobson (1931)* the i is missing in nominator on the r.h.s.). This formula can be written in our denotation (see Section 3) as

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{a\,\varepsilon} \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} \frac{(n-|m|)!}{(n+|m|)!} \cdot \\ \cdot \mathrm{p}_n^{|m|}(\kappa(\varepsilon)\,\upsilon)\; \mathrm{q}_n^{|m|}(\kappa(\varepsilon)\,\upsilon')\; \mathrm{Y}_{n,m}(\boldsymbol{v})\, \mathrm{Y}^*_{n,m}(\boldsymbol{v}'). \qquad (5.2)$$

For any function $f(\boldsymbol{v})$ that belongs to the class S we have (for $\boldsymbol{r} \in D_{\mathrm{int}}$, thus $0 \leq \upsilon < 1$)

$$\frac{1}{4\pi} \int \mathrm{d}\Xi' \, \frac{f(\boldsymbol{v}')}{|\boldsymbol{r} - \boldsymbol{s}(\boldsymbol{v}')|} = \frac{1}{a\,\varepsilon} \sum\nolimits_{n \geq 0} \sum\nolimits_{|m| \leq n} \frac{(n-|m|)!}{(n+|m|)!} \cdot \\ \cdot \, \mathrm{p}_n^{|m|}(\kappa(\varepsilon)\,\upsilon) \; \mathrm{q}_n^{|m|}(\kappa(\varepsilon)) \; f_{n,m} \, \mathrm{Y}_{n,m}(\boldsymbol{v}), \tag{5.3}$$

where the series on the r.h.s. converges according to (3.12) and (3.30) absolutely and uniformly (also with respect to υ for $0 \leq \upsilon \leq 1$). As the integral kernel $1/|\boldsymbol{s}(\boldsymbol{v}) - \boldsymbol{s}(\boldsymbol{v}')|$ is weakly singular, we have

$$\frac{1}{4\pi} \int \mathrm{d}\Xi' \, \frac{f(\boldsymbol{v}')}{|\boldsymbol{s}(\boldsymbol{v}) - \boldsymbol{s}(\boldsymbol{v}')|} = \frac{1}{4\pi} \int \mathrm{d}\Xi' \; \lim\nolimits_{\upsilon \to 1-} \; \frac{f(\boldsymbol{v}')}{|\boldsymbol{r} - \boldsymbol{s}(\boldsymbol{v}')|} = \\ = \lim\nolimits_{\upsilon \to 1-} \; \frac{1}{4\pi} \int \mathrm{d}\Xi' \, \frac{f(\boldsymbol{v}')}{|\boldsymbol{r} - \boldsymbol{s}(\boldsymbol{v}')|} = \\ = \lim\nolimits_{\upsilon \to 1-} \; \frac{1}{a\,\varepsilon} \sum\nolimits_{n \geq 0} \sum\nolimits_{|m| \leq n} \frac{(n-|m|)!}{(n+|m|)!} \cdot \\ \cdot \, \mathrm{p}_n^{|m|}(\kappa(\varepsilon)\,\upsilon) \; \mathrm{q}_n^{|m|}(\kappa(\varepsilon)) \; f_{n,m} \, \mathrm{Y}_{n,m}(\boldsymbol{v}) = \\ = \frac{1}{a\,\varepsilon} \sum\nolimits_{n \geq 0} \sum\nolimits_{|m| \leq n} \frac{(n-|m|)!}{(n+|m|)!} \, \mathrm{p}_n^{|m|}(\kappa(\varepsilon)) \; \mathrm{q}_n^{|m|}(\kappa(\varepsilon)) \; f_{n,m} \, \mathrm{Y}_{n,m}(\boldsymbol{v}). \tag{5.4}$$

On the other hand, it can be shown (see *Pohánka (1995)*, (2.15), (3.30), (3.31)) that (for any function $f(\boldsymbol{v})$ that belongs to the class S)

$$\frac{1}{4\pi} \int \mathrm{d}\Xi' \, \frac{f(\boldsymbol{v}')}{|\boldsymbol{s}(\boldsymbol{v}) - \boldsymbol{s}(\boldsymbol{v}')|} = \frac{1}{a} \sum\nolimits_{n \geq 0} \sum\nolimits_{|m| \leq n} D_{n,m} \, f_{n,m} \, \mathrm{Y}_{n,m}(\boldsymbol{v}), \tag{5.5}$$

where

$$D_{n,m} = \frac{1}{2n+1} \; {}_3\mathrm{F}_2(1/2,\; 1/2+m,\; 1/2-m;\; 3/2+n,\; 1/2-n;\; \varepsilon^2). \tag{5.6}$$

Therefore, comparing (5.4) and (5.5) we obtain the equality ($|m| \leq n$)

$$\frac{(n-|m|)!}{(n+|m|)!} \, \mathrm{p}_n^{|m|}(\kappa(\varepsilon)) \; \mathrm{q}_n^{|m|}(\kappa(\varepsilon)) = \\ = \frac{\varepsilon}{2n+1} \; {}_3\mathrm{F}_2(1/2,\; 1/2+m,\; 1/2-m;\; 3/2+n,\; 1/2-n;\; \varepsilon^2). \tag{5.7}$$

Using the definition of generalized hypergeometric series (see *Bateman* and *Erdélyi (1953)*, 4.1.1, 4.1.2) we easily derive the formula

$$\partial_\varepsilon \frac{\varepsilon}{2n+1}\, {}_3\mathrm{F}_2(1/2,\ 1/2+m,\ 1/2-m;\ 3/2+n,\ 1/2-n;\ \varepsilon^2) =$$

$$= \frac{1}{2n+1}\, {}_3\mathrm{F}_2(3/2,\ 1/2+m,\ 1/2-m;\ 3/2+n,\ 1/2-n;\ \varepsilon^2) = N_{n,m} \tag{5.8}$$

(see *Pohánka (1995)*, (3.29)), and thus

$$N_{n,m} = \frac{(n-|m|)!}{(n+|m|)!}\, \partial_\varepsilon\, \mathrm{p}_n^{|m|}(\kappa(\varepsilon))\, \mathrm{q}_n^{|m|}(\kappa(\varepsilon)) =$$

$$= -\frac{(n-|m|)!}{(n+|m|)!}\, \frac{1}{\varepsilon^2\sqrt{1-\varepsilon^2}} \Big(\partial \mathrm{p}_n^{|m|}(\kappa(\varepsilon))\, \mathrm{q}_n^{|m|}(\kappa(\varepsilon)) +$$

$$+ \mathrm{p}_n^{|m|}(\kappa(\varepsilon))\, \partial \mathrm{q}_n^{|m|}(\kappa(\varepsilon)) \Big). \tag{5.9}$$

Further, from the Wronskian formula for the Legendre functions (see *Bateman* and *Erdélyi (1953)*, 3.2.13, 1.2.15)

$$P_n^{|m|}(z)\, \partial_z Q_n^{|m|}(z) - Q_n^{|m|}(z)\, \partial_z P_n^{|m|}(z) = (-1)^m\, \frac{(n+|m|)!}{(n-|m|)!}\, \frac{1}{1-z^2} \tag{5.10}$$

we get using (3.3), (3.4)

$$\partial \mathrm{p}_n^{|m|}(u)\, \mathrm{q}_n^{|m|}(u) - \mathrm{p}_n^{|m|}(u)\, \partial \mathrm{q}_n^{|m|}(u) = \frac{(n+|m|)!}{(n-|m|)!}\, \frac{1}{u^2+1} \tag{5.11}$$

and thus

$$\partial \mathrm{p}_n^{|m|}(\kappa(\varepsilon))\, \mathrm{q}_n^{|m|}(\kappa(\varepsilon)) - \mathrm{p}_n^{|m|}(\kappa(\varepsilon))\, \partial \mathrm{q}_n^{|m|}(\kappa(\varepsilon)) = \frac{(n+|m|)!}{(n-|m|)!}\, \varepsilon^2. \tag{5.12}$$

We denote

$$\Lambda_{n,m} = \sqrt{1-\varepsilon^2}\, N_{n,m} \tag{5.13}$$

(see *Pohánka (1995)*, (4.4)), and from (5.9) and (5.12) we get ($|m| \le n$)

$$\frac{(n-|m|)!}{(n+|m|)!}\, \partial \mathrm{p}_n^{|m|}(\kappa(\varepsilon))\, \mathrm{q}_n^{|m|}(\kappa(\varepsilon)) = \frac{\varepsilon^2}{2}\, (1-\Lambda_{n,m}), \tag{5.14}$$

$$\frac{(n-|m|)!}{(n+|m|)!}\, \mathrm{p}_n^{|m|}(\kappa(\varepsilon))\, \partial \mathrm{q}_n^{|m|}(\kappa(\varepsilon)) = -\frac{\varepsilon^2}{2}\, (1+\Lambda_{n,m}). \tag{5.15}$$

Therefore, we have

$$\frac{\mathrm{q}_n^{|m|}(\kappa(\varepsilon))}{\partial \mathrm{q}_n^{|m|}(\kappa(\varepsilon))}\,\frac{\partial \mathrm{p}_n^{|m|}(\kappa(\varepsilon))}{\mathrm{p}_n^{|m|}(\kappa(\varepsilon))} = -\,\frac{1-\Lambda_{n,m}}{1+\Lambda_{n,m}} \tag{5.16}$$

and formula (4.17) can be expressed in the form of

$$U_{1;n,m} = -\,\frac{1}{1-\varepsilon^2}\,\frac{1}{1+\Lambda_{n,m}}\,G_{n,m}. \tag{5.17}$$

As it was shown in *Pohánka (1995)*, Section 4, $\Lambda_{0,0} = 1$ and all other coefficients $\Lambda_{n,m}$ (for $|m| \leq n$) are absolutely smaller than 1.

Further, in *Pohánka (1995)*, Section 5, the solution of the interior Dirichlet problem for the rotational ellipsoid was presented in the form of a surface integral. This can be used to express functions $U_k(\boldsymbol{r})$: if function $[U_k(\boldsymbol{s})]_{\mathrm{int}}$ can be expressed in the form of (4.4), and belongs to the class S, then for $\boldsymbol{r} \in D_{\mathrm{int}}$ we have

$$U_k(\boldsymbol{r}) = \frac{1}{4\pi}\int \mathrm{d}\Xi\; a^2\,\boldsymbol{o}(\boldsymbol{v})\cdot\frac{\boldsymbol{s}(\boldsymbol{v})-\boldsymbol{r}}{|\boldsymbol{s}(\boldsymbol{v})-\boldsymbol{r}|^3}\,u_k(\boldsymbol{v}), \tag{5.18}$$

where $\boldsymbol{s}(\boldsymbol{v})$ and $\boldsymbol{o}(\boldsymbol{v})$ are given by (1.6) and (1.9), and

$$u_k(\boldsymbol{v}) = \sum\nolimits_{n\geq 0}\sum\nolimits_{|m|\leq n}\frac{2}{1+\Lambda_{n,m}}\,U_{k;n,m}\,\mathrm{Y}_{n,m}(\boldsymbol{v}). \tag{5.19}$$

It may seem that the expression (4.2) requires less operations be performed than in the expression (5.18), (5.19), as the former contains only summation, while the latter contains summation and integration. However, it has to be pointed out, that functions $U_k(\boldsymbol{r})$ are to be calculated in a 3-dimensional domain (what is equivalent to a set of 2-dimensional domains). This means that by using (4.2), the summation has to be performed separately for every (calculation) point in this 3-dimensional domain, while by using (5.18) and (5.19) the summation is performed only for (a suitably dense set of) points in a 2-dimensional domain – surface S, and then the integration is performed for every (calculation) point in the 3-dimensional domain. Thus the number of required operations is approximately the same.

However, there is a substantial advantage in performing the integration according to (5.18) with respect to the summation according to (4.2). This is because in the latter case the presence of highly oscillating spherical functions requires to take into account a great number of terms in the sum. On the other hand, by the numerical integration according to (5.18), it is not necessary to sum such a number of terms, as the integral kernel in (5.18) is a smooth function decreasing with distance between the calculation and integration points.

Moreover, the solution in the form of a surface integral can be relatively easily transformed (in a certain approximation with respect to the parameter ε that is usually much smaller than 1) to a form containing no sums and no spherical functions; in this case it is necessary to perform only a single integration. For this transformation it is advantageous to use (in the case $k = 1$) formula (5.17) instead of (4.17), as the former contains a single transcendental function in denominator, while the latter contains a product of two transcendental functions (this matter will be presented in a separate paper).

Acknowledgments. The author is grateful to the Slovak Grant agency for science (grant No. 2/106394) for the partial support of this work.

References

Bateman H., Erdélyi A., 1953: Higher Transcendental Functions. New York, McGraw-Hill.

Heiskanen W., Moritz H., 1967: Physical Geodesy. San Francisco and London, W. H. Freeman and Company.

Hobson E. W., 1931: The theory of spherical and ellipsoidal harmonics. Cambridge University Press.

Pohánka V., 1993: Inverse problem of gravimetry for a spherical planetary body. Geophys. Inst. SAS, Bratislava.

Pohánka V., 1995: Solution of the Dirichlet and Neumann boundary problems for the Laplace equation for the rotational ellipsoid by the integral equation method. Contr. Geophys. Inst. SAS, **25**, 55–72.

Pohánka V., 1997: Method for the solution of the inverse problem of gravimetry for a planetary body of arbitrary shape. Contr. Geophys. Inst. SAS, **27**, 50–58.