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Formula for the characteristic solution of the inverse problem of gravimetry in the case of a planar surface

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A b s t r a c t : The solution of the inverse problem of gravimetry for a spherical planetary body obtained in [1] is transformed to the idealized case of a body represented by a halfspace.

Key words: local inverse problem, anomalous gravity field

The inverse problem of gravimetry for a spherical planetary body was treated in [1] and the method for its solution was presented briefly in [2]. In the case we are interested only in a local inverse problem (where the curvature of the surface of the body can be ignored), it is advantageous to use the planar version of this solution, which can be obtained by fixing some point of the surface of the body and letting the radius of the body tend to infinity. In this limit the surface of the body becomes a plane dividing the halfspace containing the matter (the interior domain) from the other (exterior) one. As we intend to treat only the local inverse problem, the density of matter and the gravity field (generated by this density) will be the anomalous ones.

We use the rectangular coordinate system x, y, z such that at the plane S (representing the surface of the body in the mentioned limit) it is $z = 0$ and in the interior domain $z < 0$. The connection between the density of matter $\rho(x, y, z)$ and gravity potential $V(x, y, z)$ is given by the Poisson equation

$$\Delta V(x, y, z) = 4\pi\,\kappa\,\rho(x, y, z), \tag{1}$$

where κ is the gravitational constant; in the exterior domain (for $z > 0$) it is $\rho(x, y, z) = 0$. The vertical component of the gravity field is

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$$g(x,y,z) = \partial_z V(x,y,z) \tag{2}$$

(it is equal to minus vertical component of the gravity acceleration) and its surface value is

$$g(x,y) = g(x,y,0); \tag{3}$$

the surface value of density is defined as

$$\rho(x,y) = \lim_{z\to 0-} \rho(x,y,z). \tag{4}$$

We consider here the case that the input of the inverse problem is represented by the functions $g(x,y)$ and $\rho(x,y)$; the output is the density of matter $\rho(x,y,z)$ in the interior domain (for $z < 0$). Further, we do not consider here the general solution of the inverse problem, but we restrict ourselves to the characteristic solution of this problem. This is a particular solution chosen from among the infinitely many particular solutions of the inverse problem to represent the whole class of the latter according to certain criteria of maximal smoothness and simplicity (for details, see [1], Chapter 9). The density of matter corresponding to this solution is called the characteristic density and is denoted by $\rho_{\mathrm{c}}(x,y,z)$.

The expression for the characteristic density in the planar case can be obtained from formulae (3.3) – (3.5) of [2] by performing the above mentioned limit (where we use also formulae (2.44) – (2.50) of [2]). The derivation of the resulting formula is lengthy, but simple, and therefore we omit it. The resulting formula for the characteristic density reads (for $z < 0$)

$$\begin{aligned}\rho_{\mathrm{c}}(x,y,z) = {} & \frac{1}{2\pi}\int_S \mathrm{d}S'\ E(\sqrt{(x-x')^2+(y-y')^2}, |z|)\ \rho(x',y') + \\ & + \frac{1}{4\pi\,\kappa}\ \frac{1}{2\pi}\int_S \mathrm{d}S'\ D(\sqrt{(x-x')^2+(y-y')^2}, |z|)\ g(x',y'),\end{aligned} \tag{5}$$

where $\mathrm{d}S' = \mathrm{d}x'\,\mathrm{d}y'$ is the surface element. Integral kernels $E(u,d)$ and $D(u,d)$ are given by

$$E(u,d) = C_1(u,d) - 32\,d^3\,C_4(u,d), \tag{6}$$

$$D(u,d) = 640\,d^3\,C_5(u,d), \tag{7}$$

where $(n \geq 0)$

$$C_n(u,d) = \frac{\mathrm{P}_n(d/\sqrt{u^2+d^2})}{\sqrt{u^2+d^2}^{\,n+1}}. \tag{8}$$

Inserting the expressions for the Legendre polynomials we obtain

$$E(u,d) = d\,\Big(F_0(u,d) - 4\,\Big(3\,F_2(u,d) - 30\,F_4(u,d) + 35\,F_6(u,d)\Big)\Big), \tag{9}$$

$$D(u,d) = 80\,\Big(15\,F_4(u,d) - 70\,F_6(u,d) + 63\,F_8(u,d)\Big), \tag{10}$$

where $(n \geq 0)$

$$F_n(u,d) = \frac{d^{\,n}}{\sqrt{u^2+d^2}^{\,n+3}}. \tag{11}$$

Now we transform formula (5) into the form suitable for numerical calculation. This is necessary, as integral kernels in this formula are not bounded for the depth $|z|$ approaching 0. Further, we want to avoid the integration over an infinite domain. We first transform the integration variables

$$x' = x + u\,\cos\varphi, \qquad y' = y + u\,\sin\varphi, \tag{12}$$

where $0 \leq u$, $0 \leq \varphi < 2\pi$, and obtain

$$\begin{aligned}\rho_{\mathrm{c}}(x,y,z) = &\frac{1}{2\pi}\int_S \mathrm{d}\sigma\; E(u,|z|)\;\rho(x+u\,\cos\varphi, y+u\,\sin\varphi) + \\ &+ \frac{1}{4\pi\,\kappa}\,\frac{1}{2\pi}\int_S \mathrm{d}\sigma\; D(u,|z|)\;g(x+u\,\cos\varphi, y+u\,\sin\varphi),\end{aligned} \tag{13}$$

where $\mathrm{d}\sigma = u\,\mathrm{d}u\,\mathrm{d}\varphi$. For an integral kernel $K(u,d)$ and function $f(x,y)$ we have

$$\begin{aligned}&\frac{1}{2\pi}\int_S \mathrm{d}\sigma\; K(u,d)\;f(x+u\,\cos\varphi, y+u\,\sin\varphi) = \\ &\qquad = \int_0^\infty \mathrm{d}u\; u\,K(u,d)\;\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi\; f(x+u\,\cos\varphi, y+u\,\sin\varphi) = \\ &\qquad = \int_0^\infty \mathrm{d}u\; u\,K(u,d)\;\Sigma(x,y,u) f(*,*),\end{aligned} \tag{14}$$

where operator $\Sigma(x,y,u)$ is defined by the formula

$$\Sigma(x,y,u)f(*,*) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi\; f(x+u\,\cos\varphi, y+u\,\sin\varphi). \tag{15}$$

Thus the function $\Sigma(p,q,u)f(*,*)$ is the mean value of function $f(x,y)$ on the circle with radius u and the centre at the point (p,q).

In the case that the integral kernel $K(u,d)$ is $F_n(u,d)$, we can perform the transformation of integral variable (for $d>0$)

$$u = d\,a(\tau), \qquad a(\tau) = \frac{\sqrt{1-\tau^2}}{\tau}, \tag{16}$$

where $0<\tau\le 1$; thus

$$\mathrm{d}u\; u = -\,\frac{d^2}{\tau^3}\,\mathrm{d}\tau, \qquad \frac{1}{\sqrt{u^2+d^2}} = \frac{\tau}{d}, \tag{17}$$

and we obtain

$$\int_0^\infty \mathrm{d}u\; u\,F_n(u,d)\;\Sigma(x,y,u)f(*,*) = \frac{1}{d}\int_0^1 \mathrm{d}\tau\;\tau^n\,\Sigma(x,y,d\,a(\tau))f(*,*). \tag{18}$$

Then we have from (14), (18), (9) and (10)

$$\frac{1}{2\pi}\int_S \mathrm{d}\sigma\; E(u,d)\;\rho(x+u\,\cos\varphi, y+u\,\sin\varphi) =$$
$$= \int_0^1 \mathrm{d}\tau\;\Big(1-4\,(3\,\tau^2-30\,\tau^4+35\,\tau^6)\Big)\;\Sigma(x,y,d\,a(\tau))\rho(*,*), \tag{19}$$

$$\frac{1}{2\pi}\int_S \mathrm{d}\sigma\; D(u,d)\;g(x+u\,\cos\varphi, y+u\,\sin\varphi) =$$
$$= \frac{80}{d}\int_0^1 \mathrm{d}\tau\;\Big(15\,\tau^4-70\,\tau^6+63\,\tau^8\Big)\,\Sigma(x,y,d\,a(\tau))g(*,*). \tag{20}$$

The last formula can be further transformed to remove the quantity d from the denominator on the r.h.s. In the case that function $g(x,y)$ and its first partial derivatives are bounded, it can be easily shown that the function $\Sigma(x,y,u)g(*,*)$ has bounded partial derivative with respect to variable u. Therefore we introduce operator $\Sigma'(x,y,u)$ defined by

$$\Sigma'(x,y,u)f(*,*) = \partial_u\,\Sigma(x,y,u)f(*,*) \tag{21}$$

and using the identity

$$15\,\tau^4 - 70\,\tau^6 + 63\,\tau^8 = \partial_\tau\, \tau^5\,(1-\tau^2)\,(3-7\,\tau^2) \tag{22}$$

we get by integration per partes

$$\frac{1}{2\pi}\int_S \mathrm{d}\sigma\; D(u,d)\; g(x+u\,\cos\varphi, y+u\,\sin\varphi) =$$

$$= \frac{80}{d}\int_0^1 \mathrm{d}\tau\; \Big(\partial_\tau\, \tau^5\,(1-\tau^2)\,(3-7\,\tau^2)\Big)\,\Sigma(x,y,d\,a(\tau))g(*,*) =$$

$$= -\frac{80}{d}\int_0^1 \mathrm{d}\tau\; \tau^5\,(1-\tau^2)\,(3-7\,\tau^2)\;\partial_\tau\,\Sigma(x,y,d\,a(\tau))g(*,*) =$$

$$= 80\int_0^1 \mathrm{d}\tau\; \tau^3\,(3-7\,\tau^2)\,\sqrt{1-\tau^2}\;\Sigma'(x,y,d\,a(\tau))g(*,*). \tag{23}$$

From (13), (19) and (23) we finally obtain (writing $z=-d$, $d>0$)

$$\rho_{\mathrm{c}}(x,y,-d) = \int_0^1 \mathrm{d}\tau\; \Big(1-4\,\tau^2\,(3-30\,\tau^2+35\,\tau^4)\Big)\,\Sigma(x,y,d\,a(\tau))\rho(*,*) +$$

$$+\frac{20}{\pi\,\kappa}\int_0^1 \mathrm{d}\tau\; \tau^3\,(3-7\,\tau^2)\,\sqrt{1-\tau^2}\;\Sigma'(x,y,d\,a(\tau))g(*,*). \tag{24}$$

The formula (24) for the characteristic solution of the inverse problem has the property that for input representing the gravity field of a single spherical inhomogeneity (located under the surface of the body) with a constant (difference) density (and surface density equal to zero), the resulting characteristic density has the main extremum exactly in the centre of this inhomogeneity. Another important property of this solution is the linear dependence on the input, which allows to add the contributions from several anomalous gravity fields.

Although the formula (24) was derived under the assumption that the input functions $g(x,y)$ and $\rho(x,y)$ tend to zero outside some bounded domain at the surface S, it gives reasonable results also for a wider class of these functions. Integrals in this formula are well defined for any bounded function $\rho(x,y)$ and for any bounded function $g(x,y)$ with bounded first partial derivatives. In the case that the function $g(x,y)$ is a constant, the second integral on the r.h.s. of (24) is zero. The same situation is even in the case that $g(x,y)$ is a linear function of x and y, since then $\Sigma(x,y,u)g(*,*)$ is a constant and $\Sigma'(x, y, u)g(*, *)$ is zero. Therefore, if the function $g(x, y)$ is given only in some bounded domain at the surface (what is usually the case for a local inverse problem) and it is not close to zero at the boundary of this domain, it is possible to change this funtion by subtraction of a suitably chosen linear term to achieve that it will be near zero at this boundary (and it can be put equal to zero outside this domain).

Similarly, it can be easily shown that for the function $\rho(x, y)$ equal to a constant, the first integral on the r.h.s. of (24) is equal to this constant (for any depth d). Therefore, it is possible to modify also the function $\rho(x, y)$ by subtraction of a suitably chosen constant.

Finally, the fact that the characteristic solution is a linear integral transformation of the input simplifies substantially the numerical calculation, as it is not necessary to use any iterative methods that usually require much computation time.

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